Bharath

INSTITUTE OF HIGHER EDUCATION AND RESEARCH

Declared as **Deemed-to-be-University** u/s 3 of the UGC Act, 1956

B.Tech Civil Engineering

U20MABT02 - Advanced Calculus and Complex Analysis

Course File

BHARATH INSTITUTE OF HIGHER EDUCATION AND RESEARCH
(Declared as Deemed - to - be - University under section 3 of UGC Act 1956)

School of Civil and Infrastructure Engineering

Vision and Mission of the Department

Vision

The Department of Civil Engineering is striving to become as a world class academic centre for quality education and research in diverse areas of civil engineering, with a strong social commitment.

Mission

Mission of the department is to achieve international recognition by:

M1: Producing highly competent and technologically capable professionals.

M2: Providing quality education in undergraduate and post graduate levels, with strong emphasis on professional ethics and social commitment.

M3: Developing a scholastic environment for the state – of –art research, resulting in practical applications.

M4: Undertaking professional consultancy services in specialized areas of civil engineering.

Program Educational Objectives (PEOs)

PEO1: PREPARATION

Civil Engineering Graduates are in position with the knowledge of Basic Sciences in general and Civil Engineering in particular so as to impart the necessary skill to analyze, synthesize and design civil engineering structures.

PEO2: CORE COMPETENCE

Civil Engineering Graduates have competence to provide technical knowledge, skill and also to identify, comprehend and solve problems in industry, research and academics, related to recent developments in civil and environmental engineering.

PEO3: PROFESSIONALISM

Civil Engineering Graduates are successfully work in various Industrial and Government organizations, both at the National and International level, with professional competence and ethical administrative insight so as to be able to handle critical situations and meet deadlines.

PEO4: SKILL

Civil Engineering Graduates have better opportunity to become a future researchers/ scientists with good communication skills so that they may be both good team-members and leaders with innovative ideas for a sustainable development.

PEO5: ETHICS

Civil Engineering Graduates are framed to improve their technical and intellectual capabilities through life-long learning process with ethical feeling so as to become good teachers, either in a class or to juniors in industry.

PROGRAMME OUTCOMES (POs)

On completion of B.Tech in Civil Engineering Programme, Graduates will have to

1) **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization for the solution of complex civil engineering problems

2) **Design/Development of Solutions:** Design solutions for complex civil engineering problems and design system components or processes that meet the specified needs with appropriate consideration for public health and safety, and cultural, societal, and environmental considerations.

3) **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

4) **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

5) **Problem analysis:** Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

6) **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

7) **Communication:** Communicate effectively on complex engineering activities with the engineering community and with t h e society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

8) **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

9) **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

10) **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal, and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

11) **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modelling to complex engineering activities with an understanding of the limitations.

12) **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

BHARATH INSTITUTE OF HIGHER EDUCATION AND RESEARCH

Declared as Deemed to be University under section 3 of UGC act 1956

173, Agaram Main Road, Selaiyur, Chennai – 600 073,Tamil Nadu

End Semester Examinations - May / June-2023

Regulation – 2020

Reg. No.								

Programme(s)	Batch	Term	Course Code(s)	Course Title
B. Tech – I Year (Common to all branches)	2020, 2021 & 2022	II	U20MABT02	Advanced Calculus and Complex Analysis

Time: Three Hours **Max Marks: 100**

Date: 27.07.2023 / FN

Part A – (10 x 2 = 20 Marks)
(Answer All Questions)

Q.No	Question	BL	CO
1	Evaluate the double integral $\int_1^2 \int_3^4 x(x+y)\,dx\,dy$.	U	CO1
2	Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$	U	CO1
3	If $\phi = x^2 + y - z - 1$, find grad ϕ at (1,0,0).	R	CO2
4	State Green's Theorem.	R	CO2
5	Prove that $L[e^{-at}] = \frac{1}{s+a}$	R	CO3
6	Prove that $L[Cosat] = \frac{s}{s^2+a^2}$	U	CO3
7	Find the fixed points of the transformation $w = \frac{6z-9}{z}$	U	CO4
8	Write the Milne-Thomson formula for imaginary part.	R	CO4
9	What are the poles of the function $f(z) = \frac{z}{(z+1)(z-i)}$	U	CO5
10	Classify the singularity of the function $f(z) = \frac{\sin z}{z}$.	R	CO5

Part B – (5 x 4 = 20 Marks)
(Answer All Questions)

Q.No	Question	BL	CO
11	Find the area of first quadrant of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	AP	CO1
12	Find $\nabla \bullet \vec{F}$ *and* $\nabla \times \vec{F}$ of the vector point function find $\vec{F} = xz^3\,\vec{i} - 2x^2yz\,\vec{j} + 2yz^4\,\vec{k}$ at the point.	AP	CO2
13	Using partial fraction method to find $L^{-1}\left[\frac{s}{(s-4)(s+9)}\right]$.	AP	CO3
14	If f(z) is an analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\lvert f(z)\rvert^p = p^2 \lvert f(z)\rvert^{p-2} \lvert f'(z)\rvert^2$.	U	CO4
15	Identify the nature of the singularity for $f(z) = \frac{\sin z - z}{z^3}$.	AP	CO5

Identify the Watch

Part C– (5 x 12 = 60 Marks) (Answer either (a) or (b) of each questions)			
16(a)	Evaluate the double integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$ by changing into polar coordinate and hence deduce that the value of $\int_0^\infty e^{-x^2} dx$	AP	CO1
	OR		
16(b)	Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral.	AP	CO1
17(a)	Verify Stoke's theorem for the function $\vec{F} = x^2 \vec{i} + xy \vec{j}$ taken around the square bounded by the lines $x=0, x=a, y=0, y=a$.	AP	CO2
	OR		
17(b)	Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$. Where C is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$.	AP	CO2
18(a)	Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$.	AP	CO3
	OR		
18(b)	Solve : $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$, given that y(t)=0, $\frac{dy}{dt} = 0$ when t=0 by using Laplace Transform.	AP	CO3
19(a)	Draw the image of the region whose vertices are *at (0,0), (1,0), (1,2) and (0,2)* in the z-plane under the transformation $w = (1+i)z$.	U	CO4
	OR		
19(b)	Determine the region in the w-plane in which the rectangle bounded by the lines x=0, y=0, x=2 and y=1 is mapped under the transformation w=z+2+3i.	U	CO4
20(a)	Evaluate $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (*i*) $2<\lvert z\rvert<3$, (*ii*) $\lvert z\rvert>3$.	AP	CO5
	OR		
20(b)	Determine the poles of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)} dz$, where C is $\lvert z\rvert = 3$.	AP	CO5

BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT – IV / EXAMINATIONS

U20MABTT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS

ASSIGNMENT – CLA IV

Q.NO	Answer the Question	Weightage	CO's	Bloom's Level
1	Find $L[t^2 e^{-3t} \cosh 2t]$	5	CO3	3
2	Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is the circle $\lvert z\rvert = 2$	5	CO5	3

NAME : A. ASHWINI

REG. NO : U22BM010 SECTION : B1

SUBJECT : ACCA.

SUBJECT CODE : U20MABT02.

ASSIGNMENT.

Find $L\left[t^2 e^{-3t} \cosh 2t\right]$

Soln:

$$L\left[t^2 e^{-3t} \cosh 2t\right]$$

$$L[f(t)] = L[\cosh 2t] = \frac{s}{s^2-4}$$

$$L[t^2 \cosh 2t] = (-1)^2 = \frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2-4)(1) - s(2s)}{(s^2-4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2-4) - 2s^2}{(s^2-4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2-4}{(s^2-4)^2}\right]$$

$$= \frac{(s^2-4)^2 - (-2s) - (-s^2-4)\left[2(s^2-4)(2s)\right]}{(s^2-4)}$$

$$= \frac{(s^2-4)\left[(s^2-4)(-2s) - (-s^2-4)(4s)\right]}{(s^2-4)^3}$$

$$= \frac{-2s^2 + 8s + 4s^3 + 16s}{(s^2-4)^2}$$

$$= L\left[e^{3t}\, t^2 \cosh 2t\right] = \left[\frac{2s^2+24s}{(s^2-4)^3}\right]_{s \to s+3}.$$

$$= \frac{2(s+3)^3 + 24(s+3)}{\left((s+3)^2-4\right)^3}$$

$$= \frac{2(s^2+9s^2+9s+27)+24s+72}{(s^2+6s+9-4)^3}$$

$$= \frac{2s^2+18s^2+18s+54+24s+72}{(s^2+6s+5)^3}$$

$$= \frac{2s^3+18s^2+42s+126}{(s^2+6s+5)^3}.$$

2. Evaluate $\int_c \frac{z}{(z+1)^2(z-2)}\,dz$; where c is the circle $|z| = 3$.

Soln:

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{(z-2)} + \frac{B}{(z+1)} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z-2)(z+1) + C(z-2)$$

Let $z=-1$.

$$1 = A(-1+1)^2 + B(-1-2)(-1+1) + C(-1-2)$$

$$1 = A(0) + B(-3)(0) + C(-3)$$

$$1 = -3C$$

$$C = -\frac{1}{3}$$

Let $z = 2$.

$$1 = A(2+1)^2 + B(2-2)(2+1) + C(2-2)$$

$$1 = A(3)^3$$

$$A = 1/9$$

Let $z = 0$.

$$1 = A(0+1)^2 + B(0-2)(0+1) + C(0-2)$$

$$1 = \frac{1}{9} + B(-2) + \left(-\frac{1}{3}\right)(-2)$$

$$1 = \frac{1}{9} + \frac{2}{3} - 2B$$

$$1 = \frac{7}{9} - 2B$$

$$\frac{9-7}{9} = -2B$$

$$-B = \frac{1}{9}$$

$$B = -\frac{1}{9}$$

1 L

$$\int \frac{1}{(z+1)^2 (z-2)} dz = \frac{1}{9} \int \frac{1}{z-2} dz - \frac{1}{9} \int \frac{1}{z+1} dz$$
$$- \frac{1}{3} \int \frac{1}{(z+1)^2} d$$

$\times$ by $z-1$ on both sides.

$$\int \frac{z-1}{(z+1)^2 (z-2)} dz = \frac{1}{9} \int \frac{(z-1)}{(z-2)} dz - \frac{1}{9}$$

$$\int \frac{z-1}{z+1}\,dz - \frac{1}{3}\int \frac{z-1}{z+1}\,dz.$$

$$|z| = 3$$

$$|x+iy| = 3$$

$$x^2+y^2 = 3.$$

i) $\frac{1}{9}\int \frac{z-1}{z-2}\,dz$ $\qquad \int_c \frac{f(z)}{z-z_0}\,dz.$

$\{ z_0 = 2$ outside circle $\}$

$$f(z) = z-1$$

$$f(z_0) = f(2) = 2-1 = 1$$

$$\int_c \frac{z-1}{z-2}\,dz \Rightarrow \int_c \frac{f(z)}{z-z_0}\,dz$$

$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\,(1)$$

$$= \frac{1}{9}\int \frac{z-1}{z-2}\,dz$$

$$= \frac{2}{9}\pi i \quad \text{—— } (1)$$

ii) $-\frac{1}{9}\int \frac{z-1}{z+1}\,dz$

$z - z_0 \rightarrow z+1$

$z_0 = -1$

$z_0 = -1$ lies inside circle.

$f(z) = z-1$

$f(z_0) = f(-1) = -1-1 = -2$

$$\int_C \frac{z-1}{z+1}\,dz = 2\pi i\, f(z_0) = 2\pi i(-2) = -4\pi i$$

$$-\frac{1}{9}\int \frac{z-1}{z+1}\,dz = \frac{4}{9}\pi i \quad \text{—— } (2)$$

iii) $-\frac{1}{3}\int \frac{z-1}{(z+1)^2}\,dz$

$z - z_0 = 0$

$(z+1)^2 = 0$.

$z = -1$

$$\Rightarrow \int_C \frac{z-1}{(z+1)^2}\, dz = 2\pi i \; f(-1)$$

$$= -4\pi i$$

$$-\frac{1}{3}\int \frac{z-1}{(z+2)^2}\, dz = \frac{4}{3}\pi i \quad — (3)$$

(1) + (2) + (3)

$$\left(\frac{2}{9} + \frac{4}{9} + \frac{4}{3}\right)\pi i \Rightarrow \left(\frac{2+4+12}{9}\right)\pi i$$

$$= \frac{18}{9}\pi i$$

$$= 2\pi i$$

BHARATH INSTITUTE OF HIGHER EDUCATION AND RESEARCH

Semester Examinations – (Month/Year)

School of Basic Sciences

Student Register Number :

Department : Mathematics

Program Name/Code : B.Tech/Common to all Branches

Course Name/Code : Advanced Calculus and Complex Analysis / U20MABT02

Date/Session/Time :

Instructions :

U20MABT02 – ADVANCED CALULUS AND COMPLEX ANALYSIS

QUESTION BANK

UNIT –I
MULTIPLE INTEGRAL

Part A – (10 x 2 = 20 Marks) (Answer ALL questions)				
Q.No	***Question***	***Unit***	***BL (R/U/Ap/An/E/C)***	***CO***
1	Evaluate the double integral $\int_1^2 \int_x^{x^2} x\,y\,dy\,dx$.	I	R	CO1
2	Evaluate the double integral $\int_0^1 \int_2^3 (x^2+y^2)dx\,dy$.	I	U	CO1
3	Evaluate the double integral $\int_1^2 \int_3^4 x(x+y)dx\,dy$.	I	U	CO1
4	Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\,\sqrt{1-y^2}}$.	I	U	CO1
5	Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$	I	U	CO1
6	Evaluate the double integral $\int_0^1 \int_y^{y^2} (x+y)dx\,dy$.	I	U	CO1
7	Evaluate the double integral $\int_0^{\pi} \int_0^{\sin\theta} r\,dr\,d\theta$	I	U	CO1
8	Evaluate $\iint_s x^2 y^2\,dx\,dy$ over the region bounded by the straight line $x=0,\ x=3,\ y=0,\ y=3$.	I	U	CO1
9	Evaluate $\int_1^a \int_1^b \frac{dx\,dy}{xy}$.	I	R	CO1

10	Solve the double integral $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$	I	U	CO1

Part B – (10 x 4 = 40 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
11	Find the area using double integral bounded by $y=x^2$ and $y=2x+3$.	I	Ap	CO1
12	Find the area using double integral bounded by the lines $x=0,\ y=1\ and\ y=x$	I	Ap	CO1
13	Evaluate $\iint_R (x-y)\,dx\,dy$, where R is the region bounded by the straight line $y=x$ and parabola $y=x^2$.	I	Ap	CO1
14	Find the area of first quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	I	Ap	CO1
15	Find area of the circle $x^2+y^2=a^2$ using double integral.	I	Ap	CO1
16	Find the area between $y=x$ and $y=x^2$?	I	Ap	CO1
17	Find the area between parabola $y^2=4ax$ and $x^2=4ay$.	I	Ap	CO1
18	Find the area of the cardioid $r=a(1+\cos\theta)$.	I	Ap	CO1
19	Solve the double integral $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$	I	Ap	CO1
20	Evaluate the double integral $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$	I	Ap	CO1

Part C – (5 x 12 = 60 Marks)
(Answer ALL questions)

Q.No	*Question*		Ap	*CO*
21 (a)	Change the order of integration to evaluate $\int_0^{a}\int_y^{a}\frac{x}{x^2+y^2}\,dx\,dy$.	I	Ap	CO1
	or			
21 (b)	Change the order of integration to evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\ dy\,dx$.			

22 (a)	Evaluate the double integral $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$ by changing into polar coordinate and hence deduce that the value of $\int_0^{\infty} e^{-x^2}\,dx$	I	Ap	CO1
	or			
22 (b)	Change the order of integration to evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx.$			
23 (a)	Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using triple integral.	I	Ap	CO1
	or			
23 (b)	Evaluate the triple integral $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}.$			
24(a)	Evaluate the triple integral $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}.$	I	Ap	CO1
	or			
24 (b)	Change the order of integration to evaluate $\int_0^{1}\int_{x^2}^{2-x} xy\,dy\,dx.$			
25 (a)	Change the order of integration to evaluate $\int_0^{a}\int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx.$.	I	Ap	CO1
	or			
25 (b)	Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral			

UNIT –II
VECTOR CALCULUS

Part A – (10 x 2 = 20 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
1	Find $\nabla\phi$, *if* $\phi=\log(x^2+y^2+z^2)$.	II	R	CO2
2	Find grad ϕ at (1,1,1), if $\phi=xyz$.	II	R	CO2
3	If $\phi=x^2+y-z-1$, find grad ϕ at (1,0,0).	II	R	CO2

4	Find ∇r, if $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$.	II	U	CO2
5	Find the Directional Derivative of $\phi=xy+yz+zx$ at (1,2,0) in the direction of $\vec{i}+2\vec{j}+2\vec{k}$.	II	U	CO2
6	State Green's Theorem	II	R	CO2
7	If $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, find $\nabla\times\vec{F}$.	II	U	CO2
8	If $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, find $\nabla\bullet\vec{F}$	II	R	CO2
9	Find the Unit Normal Vector to the surface $\phi=x^3-xyz+z^3-1$ at the point $(1, 1, 1)$.	II	U	CO2
10	State Stoke's Theorem	II	R	CO2

Part B – (10 x 4 = 40 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
11	Find the Unit Normal Vector to the surface $x^2-y^2+z=2$ at the point $(1, -1, 2)$.	II	Ap	CO2
12	If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, prove that $\nabla r^n=nr^{n-2}\vec{r}$.	II	Ap	CO2
13	If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, prove that $\nabla r=\frac{\vec{r}}{r}$.	II	Ap	CO2
14	Find $\nabla\bullet\vec{F}$ *and* $\nabla\times\vec{F}$ of the vector point function find $\vec{F}=xz^3\vec{i}-2x^2yz\vec{j}+2yz^4\vec{k}$ at the point.	II	Ap	CO2
15	$(1, -1, 1)$. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point (2, -1, 2).	II	Ap	CO2
16	Find the value of "a", Show that the vector $\vec{F}=(x+3y)\vec{i}+(y-2z)\vec{j}+(x+az)\vec{k}$ is solenoidal	II	Ap	CO2
17	Show that the vector $\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$, is irrotational.	II	Ap	CO2
18	Show that $\vec{F}=(y^2+2xz^2)\vec{i}+(2xy-z)\vec{j}+(2x^2z-y+2z)\vec{k}$ is irrotational.	II	Ap	CO2
19	Show that the vector $\vec{F}=2xy\vec{i}+(x^2+2yz)\vec{j}+(y^2+1)\vec{k}$ is irrotational.	II	Ap	CO2
20	Show that the vector $\vec{F}=3y^4z^2\vec{i}+4x^3z^2\vec{j}-3x^2y^2\vec{k}$ is solenoidal.	II	Ap	CO2

Part C – (5 x 12 = 60 Marks)
(Answer ALL questions)

Q.No	*Question*		*BL (R/U/Ap/An/E/*	CO2

			C)	
21(a)	Verify Green's theorem in the plane for $\int_C (3x-8y^2)dx+(4y-6xy)dy$. Where C is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$.	II	Ap	CO2
	or			
21(b)	Verify Stoke's theorem for the function $\vec{F}=x^2\ \vec{i}+xy\ \vec{j}$ taken around the square bounded by the lines $x=0,\ x=a, y=0, y=a$.			
22 (a)	Verify Green's theorem in the plane for $\int (x^2-xy^3)dx+(y^2-2xy)dy$ taken around the square bounded by the lines $x=0,\ x=2, y=0, y=2$.	II	Ap	CO2
	or			
22 (b)	Verify Green's theorem in a plane for the integral $\int_C (x-2y)dx+xdy$ taken round the circle $x^2+y^2=4$.			
23 (a)	Verify the Gauss Divergence Theorem for $\vec{F}=4xz\ \vec{i}\ y^2\ \vec{j}+yz\ \vec{k}$ over the cube bounded by $x=0,\ x=1, y=0, y=1, z=0\ z=1$.	II	Ap	CO2
	or			
23(b)	Verify Green's theorem in a plane for the integral $\int_C (x-2y)dx+xdy$ taken round the circle $x^2+y^2=1$.			
24 (a)	Verify Gauss divergence theorem for $\vec{F}=(x^2-yz)\ \vec{i}+(y^2-zx)\vec{j}+(z^2-xy)$ and S is the surface of the rectangular parallelepiped bounded by $x=0,\ x=a, y=0, y=b,\ z=0,\ z=c$.	II	Ap	CO2
	or			
24 (b)	Verify Green's theorem in a plane for $\int_C (xy+y^2)dx+x^2\,dy$, where C is the closed curve of the region bounded by $y=x\ and\ y=x^2$.			
25(a)	Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$. Where C is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$.	II	Ap	CO2
	or			
25(b)	Verify Stoke's theorem for the function $\vec{F}=(x^2-y^2)\vec{i}+2xy\ \vec{j}$ taken around the rectangle bounded by the lines $x=0,\ x=a, y=0, y=b$.			

UNIT-III
LAPLACE TRANSFORMS

Part A – (10 x 2 = 20 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/ E/C)*	*CO*
1	Prove that $L[e^{-at}] = \frac{1}{s+a}$	III	R	CO3
2	Prove that $L[Cos\ at] = \frac{s}{s^2 - a^2}$	III	R	CO3
3	State first Shifting property.	III	R	CO3
4	Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$	III	U	CO3
5	Prove that $L[Sinh\ at] = \frac{a}{s^2-a^2}$	III	R	CO3
6	Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$	III	U	CO3
7	Prove that $L[Cosh\ at] = \frac{s}{s^2-a^2}$	III	R	CO3
8	Find $L^{-1}\left[\frac{2}{s^2+9}\right]$	III	U	CO3
9	$L[f'(t)] =$ ______ ii) $L[f''(t)] =$ ______	III	R	CO3
10	Prove that $L[Cosat] = \frac{s}{s^2+a^2}$	III	U	CO3

Part B – (10 x 4 = 40 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
11	Find $L[t^2\ Cos\ 2t]$	III	Ap	CO3
12	Find $L[e^{-3t}\ Sinh\ 2t]$	III	Ap	CO3
13	Find $L[e^{3t} Sin\ 2t]$	III	Ap	CO3
14	Find $L[\ t^2\ Cosh\ 3t]$	III		CO3
15	Using partial fraction method to find $L^{-1}\left[\frac{s}{(s-4)(s+9)}\right]$.	III	Ap	CO3
16	Find $L[e^{2t} + 7Sin\ 3t + 5\ Cosh\ t]$	III	Ap	CO3
17	Find $L^{-1}\left[\frac{s+1}{s(s-2)(s+3)}\right]$ using partial fraction method.	III	Ap	CO3
18	Find $L^{-1}\left[\frac{s+2}{s^2+9}\right]$.	III	Ap	CO3
19	Find $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$.	III	Ap	CO3
20	Find $L[Sin\ 5t\ Cos\ 3t]$	III	Ap	CO3

Part C – (5 x 12 = 60 Marks) *(Answer ALL questions)*				
Q.No	*Question*		*BL (R/U/Ap/An/E/C)*	*CO*
21 (a)	Find the Laplace Transform of $f(t) = \begin{cases} k, & 0 < t < \frac{a}{2} \\ -k, & \frac{a}{2} < t < a \end{cases}$	III	Ap	CO3
	or			
21(b)	Find $L[t^2 e^{-3t} \, Cosh\, 2t]$			
22 (a)	Find the Laplace Transform of $f(t) = \begin{cases} t, & 0 < t < b \\ 2b - t, & b < t < 2b \end{cases}$ if $f(t + 2b) = f(t)$	III	Ap	CO3
	or			
22 (b)	Find $L^{-1} = \left[\frac{7S-11}{(S+1)(S-2)^2}\right]$			
23(a)	Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$.	III	Ap	CO3
	or			
23(b)	Find $L^{-1} = \left[\frac{4S^2+5S-3}{(S+1)^2(S+2)}\right]$			
24(a)	Using Laplace Transform, solve $(D + 1)y = t^2 + 3t + 2$ given that $y(t) = 0$ when t=0.	III	Ap	CO3
	or			
24 (b)	Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right]$			
25 (a)	Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+9)^2}\right]$	III	Ap	CO3
	or			
25 (b)	Solve : $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$, given that y(t)=0, $\frac{dy}{dt} = 0$ when t=0 by using Laplace Transform.			

UNIT-IV

ANALYTIC FUNCTIONS

Part A – (10 x 2 = 20 Marks) *(Answer ALL questions)*				
Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
1	Verify C-R equation of the function f(z) = $\bar{z}$.	IV	R	CO4
2	Find the invariant points of the transformation w = $\frac{3-z}{1+z}$	IV	U	CO4
3	Check whether the function f(z) = z^2 is analytic or not?	IV	U	CO4
4	Find the fixed points of the transformation $w = \frac{6z-9}{z}$	IV	U	CO4
5	Show that $v = 3x^2y - y^3$ is Harmonic function	IV	U	CO4

6	Find the fixed points of the transformation $w = \frac{2z+6}{z+7}$	IV	U	CO4
7	Write the Bilinear Transformation formula.	IV	R	CO4
8	Find the critical points of the function $f(x) = \frac{x+1}{x^2+1}$	IV	U	CO4
9	Write the Milne-Thomson formula for imaginary part.	IV	R	CO4
10	Check whether the function $f(z) = \|z\|^2$ is analytic or not?	IV	U	CO4

Part B – (10 x 4 = 40 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
11	Construct the Analytic function f(z) which the real part $e^x \cos y$	IV	U	CO4
12	Find the critical points of the function $f(x) = \frac{x+1}{x^2+1}$	IV	U	CO4
13	If f(z) is an analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\|f(z)\|^p = p^2\|f(z)\|^{p-2}\|f'(z)\|^2$..	IV	U	CO4
14	Find the bilinear transformation that maps the points $z = 0, -i, -1$ onto the points $w = i, 1, 0$ respectively.	IV	U	CO4
15	If f(z) is an analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\|f(z)\|^2 = 4\|f'(z)\|^2$.	IV	U	CO4
16	Find the bilinear transformation that maps the points $z = -1, 0, 1$ onto the points $w = -1, -i, 1$ respectively.	IV	U	CO4
17	Find the bilinear transformation that maps the points $\infty, i, 0$ in the $z-$plane onto the points $0, i, \infty$ in the $w-$plane	IV	U	CO4
18	Find the image of the region bounded by $x = 0,\ y = 0,\ x = 1$ *and* $y = 2$, under the transformation $w = z+2-i$.	IV	U	CO4
19	Find the image of the circle $\|z\| = 1$ under the transformation $w = 4z$.	IV	U	CO4
20	Find the image of the circle $\|z\| = 2$ under the transformation $w = 3z$.	IV	U	CO4

Part C – (5 x 12 = 60 Marks)
(Answer ***Either (a) or (b)*** of each question)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
21 (a)	Find the image of $\|z+1\| = 1$ under the transformation $w = 1/z$	IV	U	CO4
	or			
21(b)	Find f(z), if the Imaginary part is $e^x(x \sin y + y \cos y)$. Also			

	find its Conjugate			
22 (a)	Find the image of |z-2i| =2 under the transformation w=1/z	IV	U	CO4
	or			
22 (b)	Draw the image of the region whose vertices are *at (0,0), (1,0), (1,2) and (0,2)* in the z-plane under the transformation $w = (1+i)z$.			
23 (a)	Determine the analytic function whose real part is $v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$	IV	U	CO4
	or			
23 (b)	Find f(z), if the Real part is $e^x(x\cos y - y\sin y)$. Also find its Conjugate.			
24(a)	Determine the Analytic function f(z), whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$, Also find its conjugate	IV	U	CO4
	or			
24 (b)	Determine the region in the w-plane in which the rectangle bounded by the lines x=0, y=0, x=2 and y=1 is mapped under the transformation w=z+2+3i			
25 (a)	If $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, then find an analytic function $f(z)$.	IV	U	CO4
	or			
25 (b)	Draw the image of the square whose vertices are *at (0,0), (1,0), (1,2) and (0,2)* in the z-plane under the transformation $w = (1+i)z$.			

UNIT – V
COMPLEX INTEGRATION

Part A – (10 x 2 = 20 Marks) *(Answer ALL questions)*				
Q.No	***Question***	***Unit***	***BL (R/U/Ap/An/E/C)***	***CO***
1	Write the Cauchy's integral formula.	V	U	CO5
2	Evaluate $\int \frac{1}{z+2}\,dz$, where $\|z-1\| = \frac{1}{2}$	V	U	CO5
3	What are the poles of the function $f(z) = \frac{z}{(z+1)(z-i)}$	V	U	CO5
4	What is the singular point of the function $f(z) = \frac{1}{z+i}$.	V	R	CO5
5	Write the formula for finding the residue of a function at a pole of order 'm'.	V	R	CO5

6	Evaluate $\int_c \frac{z^2-3z+5}{z+4}dz$ where 'C' is the circle $\|z\|=2$, using Cauchy's Residue Theorem.	V	U	CO5
7	Find the residue at z = 0 for $f(z)=\cot z$	V	U	CO5
8	Classify the singularity of the function $f(z)=\frac{\sin z}{z}$.	V	R	CO5
9	Evaluate $f(z)=\frac{z}{(z+1)(z+2)}$ about z = - 2.	V	U	CO5
10	State Cauchy's residue Theorem.	V	R	CO5

Part B – (10 x 4 = 40 Marks)
(Answer ALL questions)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
11	Evaluate using Cauchy's Integral formula $\int_C \frac{z+1}{(z-3)(z-1)}dz$, where C is $\|z\|=2$.	V	Ap	CO5
12	Evaluate $\frac{1}{z-2}$ at $z=1$ in Taylor's series.	V	Ap	CO5
13	Identify the nature of the singularity for $f(z)=\frac{\sin z - z}{z^3}$.	V	Ap	CO5
14	Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}dz$, where C is the circle $\|z\|=1$.	V	Ap	CO5
15	Evaluate $f(z)=\cos z$ as a Taylor's series about the point $z=0$.	V	Ap	CO5
16	Calculate the residue of $f(z)=\frac{1-e^{2z}}{z^3}$.	V	Ap	CO5
17	Find the residue of $f(z)=\frac{z^2}{(z-1)^2}$ where C is the $\|z\|=3$.	V	Ap	CO5
18	Evaluate $f(z)=\sin z$ as a Taylor's series about the point $z=0$.	V	Ap	CO5
19	Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is the circle $\|z\|=3$.	V	Ap	CO5

20	Find the singularity of $f(z)=\dfrac{1-e^{z}}{z^{4}}$.	V	Ap	CO5

Part C – (5 x 12 = 60 Marks)
(Answer ***Either (a) or (b)*** of each question)

Q.No	*Question*	*Unit*	*BL (R/U/Ap/An/E/C)*	*CO*
21 (a)	Evaluate $\int_{-\infty}^{\infty}\dfrac{x^2\,dx}{(x^2+1)(x^2+4)}$ by using Contour Integration	V	Ap	CO5
	Or			
21 (b)	..Evaluate $\int_C \dfrac{\cos\pi z^2}{(z-1)(z-2)}dz$, where C is a circle $\lvert z\rvert=3$.			
22 (a)	Evaluate $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (*i*) $2<\lvert z\rvert<3$, (*ii*) $\lvert z\rvert>3$.	V	Ap	CO5
	or			
22 (b)	Evaluate $\int_0^{2\pi}\dfrac{d\theta}{5+3\cos\theta}$ by using Contour Integration			
23 (a)	Find the Laurent's series expansion of $f(z)=\dfrac{7z-2}{z(z-2)(z+1)}$ in $1<\lvert z+1\rvert<3$.	V	Ap	CO5
	or			
23 (b)	Evaluate $\int_0^{2\pi}\dfrac{d\theta}{13+5\sin\theta}$ by using Contour Integration.			
24 (a)	Evaluate $\int_{-\infty}^{\infty}\dfrac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ by using Contour Integration.	V	Ap	CO5
	or			
24 (b)	Determine the poles of $f(z)=\dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C\dfrac{z^2}{(z-1)^2(z+2)}dz$, where C is $\lvert z\rvert=3$.			
25 (a)	Evaluate $\int_0^{2\pi}\dfrac{d\theta}{2+\cos\theta}$ by using Contour Integration.	V	Ap	CO5
	or			
25 (b)	Evaluate $\int_C\dfrac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$, where C is $\lvert z\rvert=3$, using Cauchy's Integral formula.			

CO / Level	Remember (R)	Understand (U)	Apply (Ap)	Analyse (An)	Evaluate (E)	Create (C)	Total Marks
CO 1							
CO 2							
CO 3							
CO 4							
CO 5							
CO 6							
Total							
						Grand Total	100

Bharath
INSTITUTE OF HIGHER EDUCATION AND RESEARCH
(Declared as Deemed-to-be University under section 3 of UGC Act, 1956)
(Vide Notification No. F.9-5/2000 - U.3, Ministry of Human Resource Development, Govt. of India, dated 4th July 2002)
Accredited by NAAC
Phone : 044-22290742 / 22290125 . Telefax : 044-22293886
Website : www.bharathuniv.ac.in
173, Agaram Road, Selaiyur, Tambaram, Chennai - 600 073. Tamil Nadu.

CONSOLIDATED INTER NAL MARKS - THEORY

School:	Scicene and Humanuties
Year:	2022 - 2023
Subject Code	U20MABT02

Program: B.Tech
Section: B1
Subject Name: ACCA

Register Number	Student Name	Continuous Learning Assessment			Assignment (CLA-4)		CLA-1 (10%)	CLA-2 (15%)	CLA-3 (15%)	CLA-4 (10%)	TOTAL (50 Marks)
		CLA-1	CLA-2	CLA-3	Assign.1	Assign.2					
U22BM001	ADHISH R	18	18	20	5	5	6	9	10	10	35
U22BM002	AKKISETTI SAI VIGNESH	20	18	20	5	5	7	9	10	10	36
U22BM003	ALLAM SAI VARDHAN	20	19	18	5	5	7	10	9	10	35
U22BM004	REDDY	19	23	15	5	5	6	12	8	10	35
U22BM005	ANDE RAMA SRI	26	20	19	5	5	9	10	10	10	38
U22BM007	ANU TAYAL V	30	30	30	5	5	10	15	15	10	50
U22BM008	ARAVINDA MEERAA K	15	16	13	5	5	5	8	7	10	30
U22BM009	ARIVUMANI R	18	16	23	5	5	6	8	12	10	36
U22BM010	ASHWINI A	25	26	27	5	5	8	13	14	10	45
U22BM011	BANDAPALLI UDAY KIRAN	18	18	20	5	5	6	9	10	10	35
U22BM012	KUMAR REDDY	15	19	20	5	5	5	10	10	10	35
U22BM013	BOGGULA NAVEEN KUMAR REDDY	30	30	30	5	5	10	15	15	10	50
U22BM014	BONAGIRI SATHVIK	12	12	14	5	5	4	6	7	10	27
U22BM015	BUKKAM BUDI PREMKUMAR	30	30	30	5	5	10	15	15	10	50
U22BM016	CHEBOLU MOULI SATYASRI	22	22	20	5	5	7	11	10	10	38
U22BM017	CHILAKALAPUDI SHANKAR	22	24	22	5	5	7	12	11	10	40
U22BM018	CHILLAGORLA ISHWARYA	28	28	28	5	5	9	14	14	10	47
U22BM019	CHINDAM DIVYASREE	24	20	20	5	5	8	10	10	10	38
U22BM020	CHINNAKKA RITHWIK REDDY	27	28	28	5	5	9	14	14	10	47
U22BM021	CHITRA G	26	24	26	5	5	9	12	13	10	44

U22BM022	DANDA SAI CHARAN	29	28	29	5	5	10	14	15	10	48
U22BM023	DEVAGANI INDHU	30	30	30	5	5	10	15	15	10	50
U22BM024	DHADIGE BHANUSRI	22	19	22	5	5	7	10	11	10	38
U22BM025	K DIYA	26	26	26	5	5	9	13	13	10	45
U22BM026	DOLA KIRANMAI	22	22	21	5	5	7	11	11	10	39
U22BM027	DUDEKULA SHOYAB MALIK	20	20	20	5	5	7	10	10	10	37
U22BM028	DURGA K	30	30	30	5	5	10	15	15	10	50
U22BM029	GANTA ADI NARAYANA REDDY	22	20	22	5	5	7	10	11	10	38
U22BM030	GOMARAM DEVENDAR REDDY	17	18	20	5	5	6	9	10	10	35
U22BM031	GUBBALA BOBBY GANESH	28	28	26	5	5	9	14	13	10	46
U22BM032	GUDA NISHITHA	12	11	10	5	5	4	6	5	10	25
U22BM033	HARINI T	22	9	7	5	5	7	5	4	10	25
U22BM034	HARISH R	18	20	18	5	5	6	10	9	10	35
U22BM035	D HIMAYATH HUSSAIN	20	17	19	5	5	7	9	10	10	35
U22BM036	INAMPUDI LIKKIN BABU	10	10	19	5	5	3	5	10	10	28
U22BM037	JADA SHIVA SANKARA RAO	16	20	19	5	5	5	10	10	10	35
U22BM038	JAGILAM HARINI	21	18	17	5	5	7	9	9	10	35
U22BM039	JEREEN CHELES JOE	28	28	28	5	5	9	14	14	10	47
U22BM040	JILLEPALLI RISHI RISHI	20	20	16	5	5	7	10	8	10	35
U22BM041	JINKALA NANDINI	26	26	26	5	5	9	13	13	10	45
U22BM042	JUTLA VEERESH	18	18	22	5	5	6	9	11	10	36
U22BM043	KACHANA NIRANJAN REDDY	17	19	20	5	5	6	10	10	10	35
U22BM044	KALAKAMBAM VASANTHKUMAR	24	21	18	5	5	8	11	9	10	38
U22BM045	KALAKUNTLA SANATH RAO	14	16	10	5	5	5	8	5	10	28
U22BM047	KANDULA LAKSHMAN YASWANTH	20	20	16	5	5	7	10	8	10	35
U22BM048	KATTA MEDINI REDDY	26	AB	AB	4	4	9	5	7	8	29
U22BM049	KEERTHISREE BONGU	30	30	30	5	5	10	15	15	10	50
U22BM050	KETHIREDDY CHARMI	14	AB	AB	5	5	5	7	6	10	28
U22BM051	KOLLA YESHVANTH REDDY	21	22	22	5	5	7	11	11	10	39
U22BM052	KONDURU SAI KUMAR RAJU	27	26	24	5	5	9	13	12	10	44
U22BM053	KOTHAKALVA PRUDHVI	AB	AB	12	5	5	5	7	6	10	28
U22BM054	KUMMARI RUPAK	18	18	20	5	5	6	9	10	10	35
U22BM056	KURUBA DHARMATEJA	30	30	30	5	5	10	15	15	10	50

U22BM057	MADDALA JASWANTH	19	22	20	5	5	6	11	10	10	37
U22BM058	MADDINENI LAKSHMANSAI	19	19	18	5	5	6	10	9	10	35
U22BM059	MALLELA HARIKRISHNA	15	19	18	5	5	5	10	9	10	34
U22BM060	MANDADI MADHURI	23	23	26	5	5	8	12	13	10	42
U22BM061	MANDHA KEERTHI PRIYA	25	27	24	5	5	8	14	12	10	44
U22BM062	MANNAVA MANOGNA	15	20	19	5	5	5	10	10	10	35
U22BM064	MOHAMED IRSATH M	16	20	17	5	5	5	10	9	10	34
U22BM065	MOHAMED JASITH	20	26	20	5	5	7	13	10	10	40
U22BM066	MOHAMMAD NADEEM	25	27	25	5	5	8	14	13	10	44
U22BM067	MUHAMMED HASEEB	18	19	18	5	5	6	10	9	10	35
U22BM068	MUKESH R	18	20	18	5	5	6	10	9	10	35
U22BM069	MUKTHAPURAM MOUNIYA	24	22	22	5	5	8	11	11	10	40
U22BM070	MULINTI VISHNU VARDHAN REDDY	15	18	19	5	5	5	9	10	10	34
U22BM072	NANDIPAMU RAVI PRAKASH	28	29	28	5	5	9	15	14	10	48
U22BM073	NANDIRAJU BINDU JYOTHIKA	26	17	22	5	5	9	9	11	10	38
U22BM074	NEELAM GEETHA MAHESWARI	18	20	18	5	5	6	10	9	10	35
U22BM075	NUDURUPATI CHANDRA SEKHAR	28	28	27	5	5	9	14	14	10	47
U22BM076	PALAPARTHI BHUVANESH	AB	2	AB	5	5	5	5	7	10	27
U22BM077	PEDDAPALLE SAMPOORNA LAKSHMI	23	23	22	5	5	8	12	11	10	40
U22BM078	PEDDI KARTHIK	27	26	26	5	5	9	13	13	10	45
U22BM079	PERAM SAI PAVAN KUMAR	22	19	16	5	5	7	10	8	10	35
U22BM080	POLAM HEMA MAYURI	19	27	20	5	5	6	14	10	10	40
U22BM081	PRADEEP C	19	20	15	5	5	6	10	8	10	34
U22BM082	PRAGALAPATI NAGA SIRISHA	24	22	23	5	5	8	11	12	10	41
U22BM091	SALOMI R	17	24	24	5	5	6	12	12	10	40
U22BM120	YOKESHWARAN D	16	18	20	5	5	5	9	10	10	34

End Semester Examinations - May/June - 2023

U20MABT02 - Advanced Calculus and Complex Analysis

Part - A

1. Evaluate the double integral $\int_1^2 \int_3^4 x(x+y)\,dx\,dy$

Ans: $\frac{91}{12}$

2. Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$.

Ans: 2

3. If $\phi = x^2 + y - z - 1$, find grad ϕ at $(1,0,0)$

$\nabla\phi = \vec{i}\frac{\partial}{\partial x}\phi + \vec{j}\frac{\partial}{\partial y}\phi + \vec{k}\frac{\partial}{\partial z}\phi$

$= 2x\vec{i} + \vec{j} - \vec{k}$

at $(1,0,0)$

$= 2\vec{i} + \vec{j} - \vec{k}$

4. State Green's theorem.

Let $P(x,y)$, $Q(x,y)$ be any two continuous function with first order partial derivative exist then,

$\int P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$, Region R (on) xy-plane.

5. Prove that $L[e^{-at}] = \frac{1}{s+a}$

$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$

$\because f(t) = e^{-at}$

$L[e^{-at}] = \frac{1}{s+a}$

6. Prove that $L[\cos at] = \frac{s}{s^2+a^2}$

$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$

$f(t) = \cos at$

$L[\cos at] = \frac{s}{s^2+a^2}$

7. Find the fixed points of the transformation $w = \frac{6z-9}{z}$

$$w = f(z) = \frac{6z-9}{z}$$

$$z = \frac{6z-9}{z}$$

$$z^2 - 6z + 9 = 0$$

$z = 3, 3$ fixed points.

8. Write the milne-thomson formula for imaginary part.

When the I.P is given,

p.d.w.r. to y & x in V, we set $\frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x}$. , $V_y(z,0) = \phi_1(z,0)$

$V_x(z,0) = \phi_2(z,0)$

$$f(z) = \int \{\phi_1(z,0) + i\phi_2(z,0)\} dz$$

9. What are the poles of the function $f(z) = \frac{z}{(z+1)(z-i)}$

$z+1 = 0$, $z - i = 0$

$z = -1$, $z = i$

$\therefore$ $z = -1, i$ are the simple poles.

10. Classify the singularity of the function $f(z) = \frac{\sin z}{z}$.

$z = 0$, $\Rightarrow$ $\frac{\sin(0)}{0} = \frac{0}{0}$.

$$\frac{\phi(z)}{\psi(z)} = \frac{\cos(z)}{1}$$

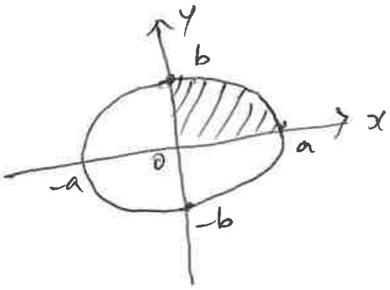
$z = 0 \Rightarrow$ $\frac{\cos(0)}{1} = \frac{1}{1} = 1$

Removable Singularity.

Part - B

11. Find the area of first quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Consider $y=0$ then $\boxed{x=a}$

$$\boxed{y=\frac{b}{a}\sqrt{a^2-x^2}}$$

$$\therefore \int_{x=0}^{x=a}\int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_0^{b/a\sqrt{a^2-x^2}}\,dx$$

$$= \frac{ab\pi}{4} \quad //$$

12. Find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$ of the vector point function find $\vec{F}=xz^3\vec{i}-2x^2yz\vec{j}+2yz^4\vec{k}$ at the point.

soln

$$\nabla\cdot\vec{F}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(xz^3\vec{i}-2x^2yz\vec{j}+2yz^4\vec{k})$$

$$=\frac{\partial}{\partial x}(xz^3)+\frac{\partial}{\partial y}(-2x^2yz)+\frac{\partial}{\partial z}(2yz^4)$$

$$\nabla\cdot\vec{F}=z^3-2x^2z+8yz^3$$

at Ex,-

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_1&F_2&F_3\end{vmatrix}=\vec{i}\,[2z^4+2x^2y]-\vec{j}\,(-3xz^2)+\vec{k}\,(-4xyz)$$

13. Using partial fraction method to find $L^{-1}\left[\frac{s}{(s-4)(s+9)}\right]$.

$$\frac{s}{(s-4)(s+9)}=\frac{A}{s-4}+\frac{B}{s+9}$$

$\boxed{A=4/13}$, $\boxed{B=9/13}$

$$=L^{-1}\left[\frac{4/13}{s-4}+\frac{9/13}{s+9}\right]$$

$$=4/13\,e^{4t}+9/13\,e^{-9t} \quad //$$

14. If $f(z)$ is an analytic function, then Prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2.$$

Consider,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = 4.\frac{\partial^2}{\partial z\partial \bar{z}} \quad \text{—(1)}$$

Multiply by $|f(z)|^p$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4\frac{\partial^2}{\partial z\partial \bar{z}}\,|f(z)|^p$$

$$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\left(|f(z)|^2\right)^{p/2} \qquad \because |z|^2 = z\bar{z}$$

$$= 4.\frac{\partial}{\partial z}\left(f(z)\right)^{p/2}\frac{\partial}{\partial \bar{z}}\left(\bar{f(\bar{z})}\right)^{p/2}$$

$$= p^2.\,|f(z)|^{p-2}.\,|f'(z)|^2 \quad //$$

15. Identify the nature of the singularity for

$$f(z) = \frac{\sin z - z}{z^3}$$

$z^3 = 0$

$f(0) = \frac{0}{0}$ form

$$g(z) = \frac{\cos z - 1}{3z^2}$$

$z = 0, \quad g(0) = \frac{0}{0}$

$$h(z) = \frac{-\sin z}{6z}$$

$z = 0, \quad h(0) = \frac{0}{0}$

$$k(z) = \frac{-\cos z}{6}$$

$z = 0 \quad k(0) = -\frac{1}{6}$

$\therefore \frac{\sin z - 1}{z^3}$ is Removable Singularity at $z = 0$.

Part-c

16 (a) Evaluate the double integral $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$ by changing into polar coordinate and hence deduce that the value of $\int_0^\infty e^{-x^2}dx$

Soln

~~First~~ let $x = r\cos\theta$, $y = r\sin\theta$

limit x: $x=0$, $x=\infty$

y: $y=0$, $y=\infty$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2=r^2 \rightarrow$ Circle with centre $(0,0)$

$\theta=0$, $\theta=\pi/2$

$r=0$, $r=\infty$.

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy \Rightarrow \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=\infty} e^{-r^2}r.dr.d\theta$$

let $r^2=t=$ $2rdr=dt$

$rdr=\frac{dt}{2}$

limit $r=0$, $t=0$

$r=\infty$, $t=\infty$

$$\int_0^{\pi/2}\left(\int_0^\infty e^{-t}\frac{dt}{2}\right)d\theta = \pi/4 \text{ . sq.unit}$$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^\infty\int_0^\infty e^{-x^2}.e^{-y^2}dxdy$$

$$\sqrt{\pi}/2 = \int_0^\infty e^{-x^2}dx.$$

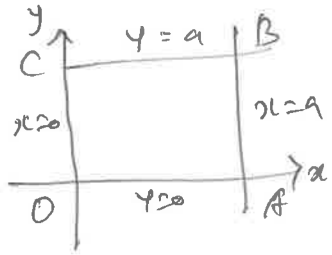
(b) Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral.

$\boxed{x=\pm a}$, $\boxed{y=\pm\sqrt{a^2-x^2}}$, $\boxed{z=\pm\sqrt{a^2-x^2-y^2}}$

$$V=\int_{x=-a}^{x=a}\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2-y^2}}^{z=\sqrt{a^2-x^2-y^2}} dz\,dy\,dx = \frac{4\pi a^3}{3} \text{ cubic units}$$

17 (a) Verify Stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ taken around the square bounded by the lines $x=0, x=a, y=0, y=a$

Soln



Stoke's theorem

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$$

R.H.S

$$\text{curl}\cdot\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = y\vec{k}$$

the surface of the rectangular in xy-plane $\vec{n} = \vec{k}$

$$= \iint y\,dx\,dy$$

$$= a^3/2 \quad \text{—} (1)$$

L.H.S

$$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

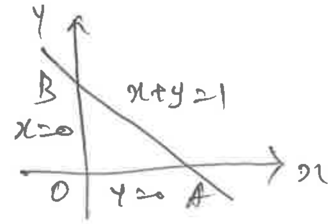
$$= a^3/3 + a^3/2 - a^3/3 + 0$$

$$= a^3/2 \quad \text{—} (2)$$

$\therefore$ L.H.S = R.H.S

(b) Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$. where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$.

Soln



Green's theorem $\int_C p\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

R.H.S $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y} 10y\,dx\,dy = 5/3 \quad \text{—} (1)$

L.H.S

$$\int_{OA} + \int_{AB} + \int_{BO}$$

$$= 3/2 + \frac{13}{6} - 2$$

$$= \frac{9+13-12}{6} = \frac{10}{6} = 5/3 \quad \text{—} (2)$$

$\therefore$ L.H.S = R.H.S

18 (a) Using Convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ ④

Soln

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t f(u)g(t-u)\,du$$

$$\cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu)\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2}$$

(b) Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$, given that $y(t)=0$, $\frac{dy}{dt}=0$ when $t=0$ by using Laplace Transform.

Soln

The given equation

$$y''(t) - 3y'(t) + 2y(t) = e^{3t}$$

Taking L.T on both sides, we get

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{3t}]$$

$$s^2L[y(t)] - sy(0) - y'(0) - 3\{sL(y(t)) - y(0)\} + 2L[y(t)] = \frac{1}{s-3}$$

$y(0)=0$ & $y'(0)=0$ and put $L[y(t)] = \bar{y}$

$$\bar{y}(s^2-3s+2) = \frac{1}{s-3}$$

$$\therefore \bar{y} = \frac{1}{(s-3)(s^2-3s+2)}$$

$$L[y(t)] = \frac{1}{(s-3)(s-1)(s-2)}$$

$B=-1$, $C=1/2$, $A=1/2$

$$= L^{-1}\left[\frac{1/2}{s-1} + \frac{-1}{s-2} + \frac{1/2}{s-3}\right]$$

$$y(t) = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}$$

19 (a) ~~Determine the~~ Draw the image of the region whose vertices are at $(0,0)$, $(1,0)$, $(1,2)$ and $(0,2)$ in the z-plane under the transformation $w = (1+i)z$.

Soln

$w = f(z)$, $w = u+iv$

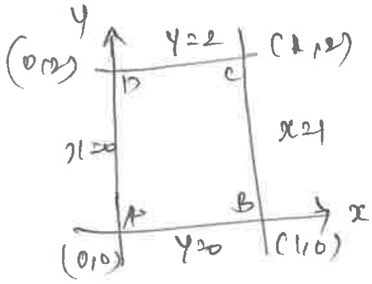
$f(z) = u+iv$, $z = x+iy$

$w = (1+i)z \Rightarrow w = (1+i)(x+iy)$

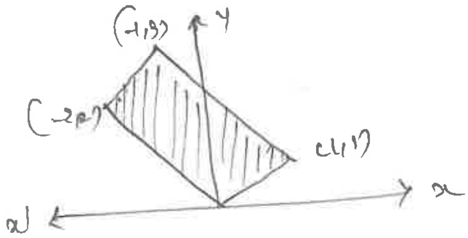
$w = x-y + i(x+y)$

$u = x-y$, $v = x+y$

point	$u = x-y$	$v = x+y$	(u,v)
$A(0,0)$	$u=0$	$v=0$	$(0,0)$
$B(1,0)$	$u=1$	$v=1$	$(1,1)$
$C(1,2)$	$u=-1$	$v=3$	$(-1,3)$
$D(0,2)$	$u=-2$	$v=2$	$(-2,2)$



z-plane



w-plane.

(b) Determine the region in the w-plane in which the rectangle bounded by the lines $x=0$, $y=0$, $x=2$, $y=1$ is mapped under the transformation $w = z+2+3i$.

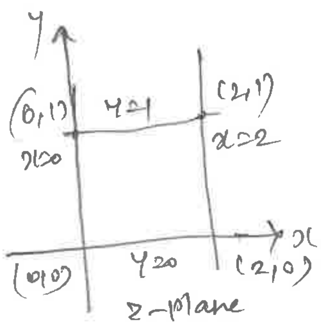
Soln

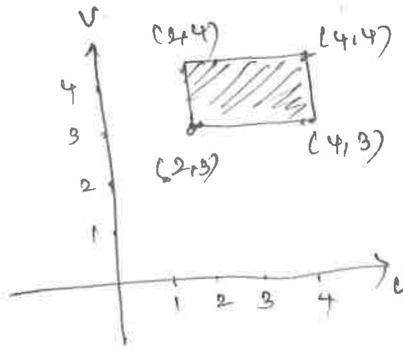
given $w = z+2+3i$

$w = u+iv$, $z = x+iy$

$u+iv = x+iy+2+3i$

$u = x+2$, $v = i(y+3)$





w-plane.

z-plane points

$(0,0)$, $(2,0)$, $(2,1)$, $(0,1)$

w-plane points

$(2,3)$, $(4,3)$, $(4,4)$, $(2,4)$

20 (a) Evaluate $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if

(i) $2<|z|<3$, (ii) $|z|>3$.

soln

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

Consider

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{A}{z+2} + \frac{B}{z+3}$$

$\boxed{A=3}$, $\boxed{B=-8}$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

By Laurent's series

(i) $2<|z|<3$

$2<|z|$

$\frac{2}{|z|}<1$

(ii) $|z|<3$

$\frac{|z|}{3}<1$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})} \Rightarrow 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n \quad //$$

(b) Determine the poles of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C \dfrac{z^2}{(z-1)^2(z+2)}\,dz$, where C is $|z|=3$

soln

To find pole,

$z-1=0$

$z=1$

$z+2=0$

$z=-2$

poles are -2 (simple pole)

$z=1$ pole of order 2

Residue of $f(z)$ at $z=-2$ simple pole $\Big\}_{a=-2}$ $= \lim_{z \to a} (z-a) f(z)$

$$= \lim_{z \to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$$

$$= 4/9 \quad \text{—} (1)$$

$z=1$ pole of order 2.

Res of $f(z)$ $\Big\}_{a=1}$ $z=1$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

$$= \frac{1}{1!} \frac{d}{dz} \left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2} (z+2)} \right]$$

$$= \frac{d}{dz} \left[\frac{z^2}{z+2} \right]$$

$$= 5/9 \quad \text{—} (2)$$

$$\int_c f(z)\, dz = 2\pi i \,[\text{Sum of Res. of } f(z)]$$

$$= 2\pi i \left[\frac{4}{9} + 5/9 \right]$$

$$= 2\pi i$$ //

BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT TEST-III

U20MABT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS
(SET- B)

Date : 23.06.2023

Academic Year / Semester : 2022-2023 / EVEN

Duration : 1 Hour 30 Minutes

Q.No.	PART – A (5 x 2 = 10)	Weightage	CO's	Bloom's Level
1	Find the fixed points of the transformation $w=\frac{6z-9}{z}$	2	CO4	2
2	What are the poles of the function $f(z)=\frac{z}{(z+1)(z-i)}$	2	CO5	1
3	Find the residue at z = 0 for $f(z)=\cot z$	2	CO5	3
4	Classify the singularity of the function $f(z)=\frac{\sin z}{z}$.	2	CO5	1
5	Evaluate $f(z)=\frac{z}{(z+1)(z+2)}$ about the pole z = - 2.	2	CO5	3
	PART – B (2 x 4 = 8)			
6	(a) Find the bilinear transformation that maps the points $z=0,\ -i,\ -1$ onto the points $w=i,\ 1,\ 0$ respectively. **(OR)** (b) Construct the Analytic function f(z) which the real part $e^x \cos y$.	4	CO4	2
7	(a) Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}dz$, where C is the circle $\lvert z\rvert=1$, Using Cauchy Integral formula. **(OR)** (b) Find the residue of $f(z)=\frac{z^2}{(z-1)^2}$ where C is the $\lvert z\rvert=3$.	4	CO5	3
	PART – C (1 x 12 = 12)			
8	(a) Find f(z), if the Real part is $e^x\left(x\cos y - y\sin y\right)$. Also find its Conjugate **(OR)** (b) Find the image of $\lvert z+1\rvert=1$ under the transformation w=1/z	12	CO4	2

CO's	Weightage
CO 4	18
CO 5	12
Total	**30**

TERM - II

U20MABT02 - ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA - III

Part-A

1. Find the fixed points of the transformation $w = \frac{6z-9}{z}$.

$z = 3, 3$

2. Find the poles of the function $f(z) = \frac{z}{(z+1)(z-i)}$

$z = -1, i$

3. Find the residue at $z=0$ for $f(z) = \cot z$

$\cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)} = \frac{\phi(z)}{\psi'(z)} = \frac{\cos(0)}{\cos(0)} = 1.$

4. Classify the singularity of the function $f(z) = \frac{\sin z}{z}$

$f(z) = \frac{\sin z}{z}$

$z=0, \Rightarrow \frac{0}{0}$ form, $\frac{\cos z}{1} = 1$

Removable singularity.

5. Evaluate $f(z) = \frac{z}{(z+1)(z+2)}$ about the pole $z=-2$.

$z=-1, z=-2$

Residue about pole $z=-2$ (simple pole)

$=2.$

Part-B

6 (A) Find the bilinear transformation that maps the points $z = 0, -i, -1$ onto the pts $w = i, 1, 0$ respectively.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$w = \frac{i(1+z)}{(1-z)}$$

(B) Construct the Analytic function $f(z)$ which the real part $e^x \cos y$.

$\phi_1(z,0) = e^z, \quad \phi_2(z,0) = 0$

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz = e^z$

$u+iv = e^{x+iy}$

$= e^x \cos y + i e^x \sin y$

By milne thomson method is

$$f(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz$$

$$= e^z \cdot z$$

To find Conjugate.

$$f(z) = u+iv$$

$$z = x+iy$$

$$u+iv = e^x (x\cos\theta - y\sin\theta) + i(e^x x\sin\theta + e^x y\cos\theta)$$

$\therefore$ Im. P : $e^x (x\sin\theta + y\cos\theta)$

(B) Find the image of $|z+1|=1$ under the transformation $w=1/z$.

$$|z+1| = 1, \quad w = 1/z$$

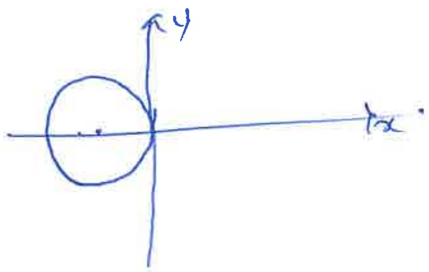
$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}$$

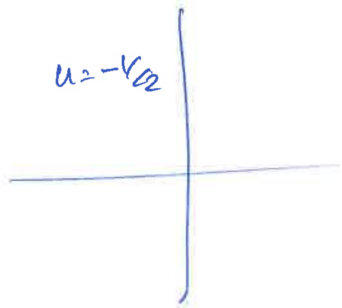
$$|z+1|=1 \Rightarrow |x+iy+1| = 1$$

$$(x+1)^2 + y^2 = 1^2$$

$$x^2 + 2x + y^2 = 0.$$

$$\boxed{u = -1/2}$$





BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT TEST-II
U20MABT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS

Date : 31.05.2023
Academic Year / Semester : 2022 – 2023 / Term-II
Maximum marks : 30 Marks
Instruction : Answer to ALL Questions

SET B

Q. No.	PART – A (5x 2 = 10)	Weightage	CO's	Bloom's Level
1	State Gauss Divergence Theorem.	2	CO2	1
2	Prove that $L[e^{-at}] = \frac{1}{s+a}$	2	CO3	2
3	Prove that $L[Sinh\ at] = \frac{a}{s^2-a^2}$	2	CO3	2
4	Find $L[1] = \frac{1}{s}$	2	CO3	2
5	Find $L[Cos\ 4t\ Cos\ 2t]$	2	CO3	2
	PART – B (2 x 4 = 8)			
6	(a) Find $L[e^{-3t} - 2cos2t + 3\ sinh3t + 5\ ]$ **(OR)** (b) Find $L[e^{-3t}(\sin 2t - \cos 3t)]$	4	CO3	3
7	(a) Find $L[e^{3t}Sin\ 2t]$ **(OR)** (b) Find $L[t^2\ e^{-3t}Sin\ 3t]$	4	CO3	3
	PART – C (1 x 12 = 12)			
8	(a) Verify Green's theorem in a plane for the integral $\int_C (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **(OR)** (b) Verify Gauss Divergence Theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the surface of the rectangular parallelepiped bounded by $x = 0,\ x = a,\ y = 0, y = b,\ z = 0, z = c$.	12	CO2	3

CO's	Weightage
CO 2	14
CO 3	16
Total	**30**

TERM – II

U20MABT02 – ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA – II

Part – A

1\. State Gauss Divergence theorem

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dv$$

2\. Prove that $L[\sinh at] = \frac{a}{s^2-a^2}$

$$\sinh at = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$L[\sinh at] = \frac{1}{2} L[e^{\theta} - e^{-\theta}] = \frac{a}{s^2 \cancel{+} a^2}$$

3\. Prove that $L[e^{-at}] = \frac{1}{s+a}$

$$L[e^{-at}] = \frac{1}{s+a}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = \frac{1}{-(s+a)}(e^{-\infty} - e^{0})$$

$$= \frac{1}{s+a} \,//$$

4\. Find $L[1] = \frac{1}{s}$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt = \frac{1}{s}$$

5\. Find $L[\cos 4t \cos 2t]$

$$\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$= \frac{s}{2}\left[\frac{1}{s^2+36} + \frac{1}{s^2+4}\right]$$

Part - C

8. Verify Green's theorem in a plane for the integral $\int_C (xy+y^2)\,dx + x^2\,dy$, where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy \rightarrow ①$$

$P = xy+y^2$ $\quad$ $Q = x^2$

$P_y = x+2y$ $\quad$ $Q_x = 2x$.

$$\iint_R (Q_x - P_y)\,dx\,dy = \frac{-1}{20}$$

$$\int P\,dx + Q\,dy = \int_C (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{C_1: y=x^2} (xy+y^2)\,dx + x^2\,dy + \int_{C_2: y=x} (xy+y^2)\,dx + x^2\,dy$$

$I = 19/20$

$II = -1/20$

$\therefore$ $I + II = -1/20$.

$\therefore$ L.H.S = R.H.S

8. Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the surface of the rectangular parallelepiped bounded by $x=0, x=a,$ $y=0, y=b, z=0, z=c$.

By Gauss Divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT TEST –I
U20MABT02- ADVANCED CALCULUS AND COMPLEX ANALYSIS
SET A

Date : 25.04.2023
Academic Year/Semester : 2022-2023/EVEN
Duration :1:30 Hours

Q.No.	PART A (5 X 2)= 10	Weightage	CO's	Bloom's Level
1	Evaluate the double integral $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$.	2	CO1	2
2	Evaluate the double integral $\int_0^1 \int_y^{y^2} (x+y)\,dx\,dy$.	2	CO1	2
3	Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$	2	CO1	2
4	If $\phi = x^2 + y - z - 1$, find grad ϕ at (1,0,0).	2	CO2	2
5	Find the Directional Derivative of $\phi = xyz$ at (1,1,1) in the direction of $\vec{i} + \vec{j} + \vec{k}$.	2	CO2	3
	PART B (2 X 4)= 8			
6	Find the area using double integral bounded by the lines $x=0,\ y=1$ *and* $y=x$ **(or)** Find the area using double integral bounded by $y=x^2$ and $y=2x+3$.	4	CO1	3
7	If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\nabla r = \frac{\vec{r}}{r}$. **(or)** Find the Unit Normal Vector to the surface $x^2 - y^2 + z = 2$ at the point $(1, -1, 2)$.	4	CO2	3
	PART C (1 X 12)= 12			
8	Evaluate the double integral $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar coordinate and hence deduce that the value of $\int_0^{\infty} e^{-x^2}\,dx$ **(or)** Change the order of integration to evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$.	12	CO1	3

CO's	Weightage
CO1	22
CO2	08
Total	30

TERM – II

U20MAB702 – ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA – I

Part – A

1. Evaluate the double integral. $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$.

Ans: $8/3$

2. Evaluate the double integral $\int_0^1 \int_y^{y^2} (x+y)\,dx\,dy$.

Ans: $-3/20$.

3. Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$.

Ans: 2

4. If $\phi = x^2+y-z-1$, find grad ϕ at $(1,0,0)$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

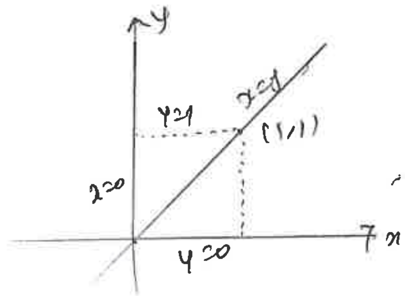
$\therefore \nabla\phi = 2\vec{i} + \vec{j} - \vec{k}$

5. Find the Directional Derivative of $\phi = xyz$ at $(1,1,1)$ in the direction of $\vec{i}+\vec{j}+\vec{k}$.

$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \sqrt{3}$.

Part – B

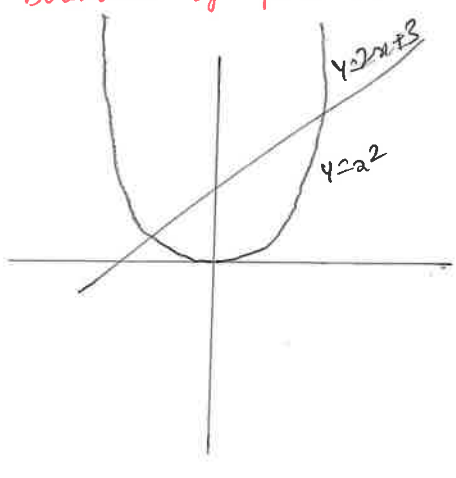
6 (A) Find the area using double integral bounded by the lines $x=0$, $y=1$ and $y=x$.



$\iint_R dx\,dy = \int_{x=0}^{1} \int_{y=0}^{x} dy\,dx$

$= \frac{1}{2}$ sq. units

(B) Find the area using double integral bounded by $y=x^2$ and $y=2x+3$.



$\boxed{x=-1,\ x=3}$, $\boxed{y=1,\ y=9}$

$\int_{-1}^{3} \int_{x^2}^{2x+3} [dy]\,dx = \int_{-1}^{3} (y)_{x^2}^{2x+3}\,dx$

$= 32/6$ sq. units

$\theta = 0, \quad \theta = \pi/2$

$r = 0, \quad r = \infty.$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\,dy$$

$r^2 = t \Rightarrow 2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

limit

$r = 0, \; t = 0$

$r = \infty, \; t = \infty.$

$$\int_0^{\pi/2} \int_0^\infty \left(e^{-t} \frac{dt}{2}\right) d\theta$$

$$= \frac{1}{2} (\theta)_0^{\pi/2} = \frac{\pi}{4} \text{ sq. units}$$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\,dy = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\, dx\,dy$$

$$\sqrt{\frac{\pi}{4}} = \int_0^\infty e^{-x^2}\, dx$$

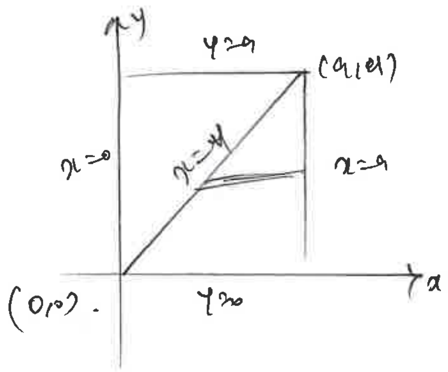
$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2}\, dx$$

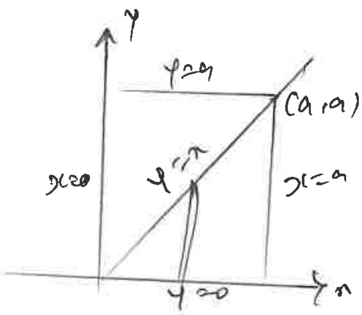
B) Change the Order of integration & evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

limit

$x = y, \quad x = a$

$y = 0, \quad y = a$





BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT – IV / EXAMINATIONS

U20MABTT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS

ASSIGNMENT – CLA IV

Q.NO	Answer the Question	Weightage	CO's	Bloom's Level
1	Find $L[t^2 e^{-3t} \cosh 2t]$	5	CO3	3
2	Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is the circle $\|z\| = 2$	5	CO5	3

NAME : G. CHITRA.

REG.NO : U22BM021 SECTION: B1

SUBJECT : ACCA.

SUBJECT CODE : U20MABT02.

ASSIGNMENT.

1. find $L\left[t^2 e^{-3t} \cosh at\right]$.

Solution:

$$L\left[t^2 e^{-3t} \cosh 2t\right].$$

$$L[f(t)] = L\left[\cosh at\right] = \frac{s}{s^2-4}$$

$$L\left[t^2 \cosh at\right] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right)$$

$$= \frac{d}{ds}\left[\frac{(s^2-4)(1) - s(2s)}{(s^2-4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2-4) - 2s^2}{(s^2-4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-s^2-4}{(s^2-4)^2}\right]$$

$$= \frac{(s^2-4)^2 \cdot (-2s) - (-s^2-4)\left[2(s^2-4)(2s)\right]}{(s^2-4)}$$

$$= \frac{2s^3 + 18s^2 + 42s + 126}{(s^2+6s+5)^3} \quad //$$

2. Evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)} \, dz$, where c is the circle $|z| = 3$.

Solution:

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{(z-2)} + \frac{B}{(z+1)} + \frac{C}{(z+1)^2}$$

$$1 = A(z+1)^2 + B(z-2)(z+1) + C(z-2)$$

Let $z = -1$

$$1 = A(-1+1)^2 + B(-1-2)(-1+1) + C(-1-2)$$

$$1 = A(0) + B(-3)(0) + C(-3)$$

$$1 = -3C$$

$$\boxed{C = \frac{-1}{3}}$$

$$\int \frac{1}{(z+1)^2(z-2)}\,dz = \frac{1}{9}\int \frac{1}{z-2}\,dz - \frac{1}{9}\int \frac{1}{z+1}\,dz$$

$$-\frac{1}{3}\int \frac{1}{(z+1)^2}\,dz$$

$\times$ by $z-1$ on both sides.

$$\int \frac{z-1}{(z+1)^2(z-2)}\,dz = \frac{1}{9}\int \frac{(z-1)}{(z-2)}\,dz - \frac{1}{9}$$

$$\int \frac{z-1}{z+1}\,dz - \frac{1}{3}\int \frac{z-1}{z+1}\,dz.$$

$$|z| = 3$$

$$|x+iy| = 3$$

$$x^2+y^2 = 3.$$

(i) $\frac{1}{9}\int \frac{z-1}{z-2}\,dz$ $\quad \int_c \frac{f(z)}{z-z_0}\,dz.$

$\boxed{z_0 = 2}$ outside circle

(iii) $\frac{-1}{3} \int \frac{z-1}{(z+1)^2} \, dz$

$z - z_0 = 0$

$(z+1)^2 = 0$

$z+1) = 0$

$\boxed{z = -1}$

$\cancel{\frac{-1}{3} \int \frac{z+1}{(z+2)^2}} \Rightarrow \int_C \frac{z-1}{(z+1)^2} \, dz = 2\pi i \, f(-1)$

$= -4\pi i$

$\frac{-1}{3} \int \frac{z-1}{(z+2)^2} \, dz = \frac{4}{3} \pi i \rightarrow 3$

$1 + 2 + 3$

$\left(\frac{2}{9} + \frac{4}{9} + \frac{4}{3} \right) \pi i \Rightarrow \left(\frac{2+4+12}{9} \right) \pi i$

$= \frac{18}{9} \pi i$

$\boxed{2\pi i}$

QUESTION BANK

Department : **MATHEMATICS**
Program Name/Code : **B TECH/** Common to all Branches
Course Name/Code : Advanced Calculus and Complex Analysis / U20MABT02

Q.No	Question	Weightage	CO	Bloom's Level
UNIT I MULTIPLE INTEGRAL				
PART – A				
1	Evaluate the double integral $\int_1^2 \int_3^4 x(x+y)\,dx\,dy$.	2	CO1	2
2	Evaluate the double integral $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$.	2	CO1	2
3	Evaluate the double integral $\int_1^2 \int_x^{x^2} x\,y\,dy\,dx$.	2	CO1	2
4	Evaluate the double integral $\int_0^1 \int_y^{y^2} (x+y)\,dx\,dy$.	2	CO1	2
5	Find the value of $\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$	2	CO1	2
6	Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\,\sqrt{1-y^2}}$.	2	CO1	2
7	Evaluate $\int_1^a \int_1^b \frac{dx\,dy}{xy}$.	2	CO1	2
8	Evaluate $\iint_s x^2y^2\,dx\,dy$ over the region bounded by the straight line $x=0,\ x=3, y=0,\ y=3$.	2	CO1	3
9	Evaluate the double integral $\int_0^{\pi} \int_0^{\sin\theta} r\,dr\,d\theta$	2	CO1	2
10	Solve the double integral $\int_0^{\pi} \int_0^{a} r\,dr\,d\theta$	2	CO1	2
PART – B				
1	Evaluate $\iint_R (x-y)\,dx\,dy,$ where R is the region bounded by the straight line $y=x$ and parabola $y=x^2$.	4	CO1	3

2	Find the area using double integral bounded by the lines $x=0,\ y=1\ and\ y=x$	4	CO1	3
3	Find the area using double integral bounded by $y=x^2$ and $y=2x+3$.	4	CO1	3
4	Find the area between $y=x$ and $y=x^2$?	4	CO1	3
5	Find area of the circle $x^2+y^2=a^2$ using double integral.	4	CO1	3
6	Find the area of first quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	4	CO1	3
7	Find the area between parabola $y^2=4ax$ and $x^2=4ay$.	4	CO1	3
8	Evaluate the double integral $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$	4	CO1	3
9	Solve the double integral $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$	4	CO1	3
10	Find the area of the cardioid $r=a(1+\cos\theta)$.	4	CO1	3
	PART – C			
1	Evaluate the double integral $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy$ by changing into polar coordinate and hence deduce that the value of $\int_0^{\infty} e^{-x^2}dx$	12	CO1	3
2	Change the order of integration to evaluate $\int_0^{a}\int_y^{a}\frac{x}{x^2+y^2}\,dx\,dy$.	12	CO1	3
3	Change the order of integration to evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\ dy\,dx$.	12	CO1	3
4	Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using triple integral.	12	CO1	3
5	Evaluate the triple integral $\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$.	12	CO1	3

6	Change the order of integration to evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$.	12	CO1	3
7	Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral.	12	CO1	3

Q.No	Question	Weightage	CO	Bloom's Level
	UNIT II **VECTOR CALCULUS**			
	PART – A			
1	If $\phi=x^2+y-z-1$, find grad ϕ at (1,0,0).	2	CO2	2
2	Find grad ϕ at (1,1,1), if $\phi=xyz$.	2	CO2	1
3	Find $\nabla\phi$, *if* $\phi=\log(x^2+y^2+z^2)$.	2	CO2	2
4	Find $\nabla \vec{r}$, *if* $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$.	2	CO2	2
5	Find the Directional Derivative of $\phi=xyz$ at (1,1,1) in the direction of $\vec{i}+\vec{j}+\vec{k}$.	2	CO2	2
6	Find the Directional Derivative of $\phi=xy+yz+zx$ at (1,2,0) in the direction of $\vec{i}+2\vec{j}+2\vec{k}$.	2	CO2	2
7	*If* $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, find $\nabla\bullet\vec{F}$	2	CO2	2
8	*If* $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, find $\nabla\times\vec{F}$.	2	CO2	2
9	Find the Unit Normal Vector to the surface $\phi=x^3-xyz+z^3-1$ at the point $(1, 1, 1)$.	2	CO2	2
10	State Green's Theorem	2	CO2	1
	PART – B			
1	If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, prove that $\nabla r=\frac{\vec{r}}{r}$.	4	CO2	3
2	If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, prove that $\nabla r^n=nr^{n-2}\vec{r}$.	4	CO2	3
3	Find the Unit Normal Vector to the surface $x^2-y^2+z=2$ at the point $(1,-1,2)$.	4	CO2	3
4	Find the angle between the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at the point (2, -1, 2).	4	CO2	3
5	Find $\nabla\bullet\vec{F}$ *and* $\nabla\times\vec{F}$ of the vector point function find $\vec{F}=xz^3\vec{i}-2x^2yz\vec{j}+2yz^4\vec{k}$ at the point $(1,-1,1)$.	4	CO2	3
6	Find the value of "a", Show that the vector $\vec{F}=(x+3y)\vec{i}+(y-2z)\vec{j}+(x+az)\vec{k}$ is solenoidal	4	CO2	3

7	Show that the vector $\vec{F}=3y^4z^2\vec{i}+4x^3z^2\vec{j}-3x^2y^2\vec{k}$ is solenoidal.	4	CO2	3
8	Show that $\vec{F}=(y^2+2xz^2)\vec{i}+(2xy-z)\vec{j}+(2x^2z-y+2z)\vec{k}$ is irrotational.	4	CO2	3
9	Show that the vector $\vec{F}=2xy\,\vec{i}+(x^2+2yz)\vec{j}+(y^2+1)\vec{k}$ is irrotational.	4	CO2	3
10	Show that the vector $\vec{F}=yz\,\vec{i}+zx\,\vec{j}+xy\,\vec{k}$, is irrotational.	4	CO2	3
PART – C				
1	Verify Green's theorem in the plane for $\int_C (3x-8y^2)dx+(4y-6xy)dy$. Where C is the boundary of the region defined by $x=0,\ y=0,\ x+y=1$.	12	CO2	3
2	Verify Stoke's theorem for the function $\vec{F}=(x^2-y^2)\vec{i}+2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x=0,\ x=a, y=0, y=b$.	12	CO2	3
3	Verify Green's theorem in a plane for $\int_C (xy+y^2)dx+x^2\,dy$, where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.	12	CO2	3
4	Verify Gauss divergence theorem for $\vec{F}=(x^2-yz)\,\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)$ and S is the surface of the rectangular parallelepiped bounded by $x=0,\ x=a, y=0, y=b,\ z=0,\ z=c$.	12	CO2	3
5	Verify Green's theorem in a plane for the integral $\int_C (x-2y)dx+xdy$ taken round the circle $x^2+y^2=1$.	12	CO2	3
6	Verify the Gauss Divergence Theorem for $\vec{F}=4xz\,\vec{i}\ y^2\,\vec{j}+yz\,\vec{k}$ over the cube bounded by $x=0,\ x=1, y=0, y=1, z=0\ z=1$.	12	CO2	3
7	Verify Stoke's theorem for the function $\vec{F}=x^2\,\vec{i}+xy\,\vec{j}$ taken around the square bounded by the lines $x=0,\ x=a, y=0, y=a$.	12	CO2	3

Q.No	Question	Weightage	CO	Bloom's Level
	UNIT III LAPLACE TRANSFORMS			
	PART – A			
1	Prove that $L[e^{at}] = \frac{1}{s-a}$	2	CO3	1
2	Prove that $L[Sinat] = \frac{a}{s^2+a^2}$	2	CO3	2
3	State first Shifting property.	2	CO3	1
4	Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$	2	CO3	2
5	Prove that $L[Sinh\ at] = \frac{a}{s^2-a^2}$	2	CO3	2
6	Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$	2	CO3	2
7	Prove that $L[Cosh\ at] = \frac{s}{s^2-a^2}$	2	CO3	2
8	Prove that $L[Cosat] = \frac{s}{s^2+a^2}$	2	CO3	2
9	$L[f'(t)] =$ ______ ii) $L[f''(t)] =$ ______	2	CO3	1
10	Find $L^{-1}\left[\frac{2}{s^2+9}\right]$	2	CO3	2
	PART – B			
1	Find $L[e^{3t}Sin\ 2t]$	4	CO3	3
2	Find $L[\ t^2\ Cosh\ 3t]$	4	CO3	3
3	Find $L[t^2\ Cos\ 2t]$	4	CO3	3
4	Find $L[e^{-3t}\ Sinh\ 2t]$	4	CO3	3
5	Find $L[e^{2t} + 7Sin\ 3t + 5\ Cosh\ t]$	4	CO3	3
6	Using partial fraction method to find $L^{-1}\left[\frac{s}{(s-4)(s+9)}\right]$	4	CO3	3
7	Find $L^{-1}\left[\frac{s+1}{s(s-2)(s+3)}\right]$ using partial fraction method.	4	CO3	3
8	Find $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$	4	CO3	3
9	Find $L[Sin\ 5t\ Cos\ 3t]$	4	CO3	3
10	Find $L^{-1}\left[\frac{s+2}{s^2+9}\right]$	4	CO3	3
	PART – C			
1	Find the Laplace Transform of $f(t) = \begin{cases} t & ,0 < t < b \\ 2b - t, & b < t < 2b \end{cases}$ if $f(t + 2b) = f(t)$	12	CO3	3
2	Find $L[t^2e^{-3t}\ Cosh\ 2t]$	12	CO3	3
3	Find the Laplace Transform of $f(t) = \begin{cases} k\ , & 0 < t < \frac{a}{2} \\ -k, & \frac{a}{2} < t < a \end{cases}$	12	CO3	3
4	Find $L^{-1} = \left[\frac{4S^2+5S-3}{(S+1)^2(S+2)}\right]$	12	CO3	3

5	Solve : $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$, given that y(t)=0, $\frac{dy}{dt} = 0$ when t=0 by using Laplace Transform.	12	CO3	3
6	Find $L^{-1} = \left[\frac{7S-11}{(S+1)(S-2)^2}\right]$	12	CO3	3
7	Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$.	12	CO3	3

.No	Question	Weightage	CO	Bloom's Level
	UNIT IV **ANALYTIC FUNCTIONS**			
	PART – A			
1	Check whether the function f(z) = z^2 is analytic or not?	2	CO4	2
2	Find the fixed points of the transformation $w = \frac{6z-9}{z}$	2	CO4	2
3	Verify C-R equation of the function f(z) = $\bar{z}$.	2	CO4	1
4	Find the invariant points of the transformation w = $\frac{3-z}{1+z}$	2	CO4	2
5	Check whether the function $f(z) = \|z\|^2$ is analytic or not?	2	CO4	2
6	Find the fixed points of the transformation w = $\frac{2z+6}{z+7}$	2	CO4	2
7	Write the Milne-Thomson formula for imaginary part.	2	CO4	1
8	Show that $v = 3x^2y - y^3$ is Harmonic function	2	CO4	2
9	Write the Bilinear Transformation formula.	2	CO4	1
10	Find the critical points of the function f(x) = $\frac{x+1}{x^2+1}$	2	CO4	2
	PART – B			
1	Find the bilinear transformation that maps the points $z = 0, -i, -1$ onto the points $w = i, 1, 0$ respectively.	4	CO4	2
2	Find the bilinear transformation that maps the points $\infty, i, 0$ in the $z-$plane onto the points $0, i, \infty$ in the $w-$pl	4	CO4	2
3	If f(z) is an analytic function, then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\|f(z)\|^p = p^2\|f(z)\|^{p-2}\|f'(z)\|^2$..	4	CO4	2
4	Construct the Analytic function f(z) which the real part $e^x \cos y$	4	CO4	2
5	Find the bilinear transformation that maps the points $z = -1, 0, 1$ onto the points $w = -1, -i, 1$ respectively.	4	CO4	2
6	If f(z) is an analytic function, then prove that	4	CO4	2

	$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\lvert f(z)\rvert^2=4\lvert f'(z)\rvert^2$.			
7	Show that an analytic function with constant real part is also constant.	4	CO4	2
8	Find the image of the circle $\lvert z\rvert = 2$ under the transformation $w = 3z$.	4	CO4	2
9	Find the image of the circle $\lvert z\rvert = 1$ under the transformation $w = 4z$.	4	CO4	2
10	Find the image of the region bounded by $x=0,\ y=0,\ x=1$ *and* $y=2$, under the transformation $w = z+2-i$.	4	CO4	2
	PART – C			
1	Find f(z), if the Imaginary part is $e^x(x\sin y+y\cos y)$. Also find its Conjugate	12	CO4	2
2	Find the image of |z+1| =1 under the transformation w=1/z	12	CO4	2
3	Find f(z), if the Real part is $e^x(x\cos y-y\sin y)$. Also find its Conjugate.	12	CO4	2
4	Draw the image of the region whose vertices are *at (0,0), (1,0), (1,2) and (0,2)* in the z-plane under the transformation $w = (1+i)z$.	12	CO4	2
5	Determine the Analytic function f(z), whose real part is $x^3-3xy^2+3x^2-3y^2+1$, Also find its conjugate	12	CO4	2
6	Find the image of |z-2i| =2 under the transformation w=1/z	12	CO4	2
7	If $u=\frac{\sin 2x}{\cosh 2y+\cos 2x}$, then find an analytic function $f(z)$.	12	CO4	2

Q.No	Question	Weightage	CO	Bloom's Level
	UNIT V COMPLEX INTEGRATION			
	PART – A			
1	Write the Cauchy's integral formula for Derivatives.	2	CO5	1
2	What are the poles of the function $f(z)=\frac{z}{(z+1)(z-i)}$	2	CO5	2
3	Evaluate $\int \frac{1}{z+2}\,dz$, where $\lvert z-1\rvert=\frac{1}{2}$	2	CO5	2
4	What is the singular point of the function $f(z)=\frac{1}{z+i}$.	2	CO5	2
5	Write the formula for finding the residue of a function at a pole of order 'm'.	2	CO5	1

6	Evaluate $\int_c \frac{z^2-3z+5}{z+4}dz$ where 'C' is the circle $\lvert z\rvert=2$, using Cauchy's Residue Theorem.	2	CO5	2
7	Find the residue at z = 0 for $f(z)=\cot z$	2	CO5	2
8	Classify the singularity of the function $f(z)=\frac{\sin z}{z}$.	2	CO5	2
9	Evaluate $f(z)=\frac{z}{(z+1)(z+2)}$ about z = - 2.	2	CO5	2
10	State Cauchy's residue Theorem.	2	CO5	1
	PART – B			
1	Evaluate using Cauchy's Integral formula $\int_C \frac{z+1}{(z-3)(z-1)}dz$, where C is $\lvert z\rvert=2$.	4	CO5	3
2	Evaluate $\frac{1}{z-2}$ at $z=1$ in Taylor's series.	4	CO5	3
3	Identify the nature of the singularity for $f(z)=\frac{\sin z-z}{z^3}$.	4	CO5	3
4	Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}dz$, where C is the circle $\lvert z\rvert=1$.	4	CO5	3
5	Evaluate $f(z)=\cos z$ as a Taylor's series about the point $z=0$.	4	CO5	3
6	Calculate the residue of $f(z)=\frac{1-e^{2z}}{z^3}$.	4	CO5	3
7	Find the residue of $f(z)=\frac{z^2}{(z-1)^2}$ where C is the $\lvert z\rvert=3$.	4	CO5	3
8	Evaluate $f(z)=\sin z$ as a Taylor's series about the point $z=0$.	4	CO5	3
9	Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is the circle $\lvert z\rvert=2$.	4	CO5	3
10	Find the singularity of $f(z)=\frac{1-e^z}{z^4}$.	4	CO5	3
	PART – C			
1	Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz$, where C is a circle $\lvert z\rvert=3$.	12	CO5	3
2	Evaluate $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$ by using Contour Integration..	12	CO5	3
3	Evaluate $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (i) $2<\lvert z\rvert<3$, (ii) $\lvert z\rvert>3$.	12	CO5	3

4	Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ by using Contour Integration	12	CO5	3
5	Find the Laurent's series expansion of $f(z)=\frac{7z-2}{z(z-2)(z+1)}$ in $1<\lvert z+1\rvert<3$.	12	CO5	3
6	Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz$, where C is $\lvert z\rvert=3$, using Cauchy's Integral formula.	12	CO5	3
7	Determine the poles of $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)}dz$, where C is $\lvert z\rvert=3$.	12	CO5	3

BHARATH INSTITUTE OF HIGHER EDUCATION AND RESEARCH

I YEAR, B.TECH, 2022 - 2023, TERM I

SECTION A1

Subject Name : Advanced calculus and Complex Analysis

SL NO.	REG.NO	STUDENT NAME	CLA 1 (10)	CLA 2 (15)	CLA 3 (15)	CLA 4 (10)	TOTAL (50)
1	U22AE001	AKELLA PRAKASH	8	13	11	10	**42**
2	U22AE002	ASHWINTH A	9	13	14	10	**46**
3	U22AE003	CHIKATE SUMEDHA ISHWAR	8	15	13	10	**46**
4	U22AE004	CHINTHA HARIKA	10	15	15	10	**50**
5	U22AE005	GULIPELLI VISHWA TEJA	8	9	10	10	**37**
6	U22AE006	GOPI M	7	12	11	10	**40**
7	U22AE007	GURRAM ARUN KUMAR	8	12	13	10	**43**
8	U22AE008	HIRTHICK N	9	13	14	10	**46**
9	U22AE009	S ISWARYA	10	15	15	10	**50**
10	U22AE010	JANNATUL NAYEM KOLY	8	11	12	10	**41**
11	U22AE011	JAYASHREE M	8	15	13	10	**46**
12	U22AE012	KAVIPRIYA A	6	14	14	10	**44**
13	U22AE013	KEDRAKA PRAHALADHUDU	9	14	15	10	**48**
14	U22AE014	KOJJA NARENDRA	10	15	15	10	**50**
15	U22AE015	KOLIKAR THARUN SAI	7	7	13	10	**37**
16	U22AE016	KOTHAMANGALA K S HARIKA	10	15	15	10	**50**
17	U22AE017	KUNCHAM BHAVYA	10	15	14	10	**49**
18	U22AE018	MADIREDDY BHARAT KUMAR	5	8	9	10	**32**
19	U22AE020	MUGENTH RASHID M R	6	13	13	10	**42**
20	U22AE021	MURALI KRISHNA	5	13	15	10	**43**
21	U22AE022	NEHA JANA	10	15	15	10	**50**
22	U22AE023	NERELLA LOKESH BABU	8	13	14	10	**45**
23	U22AE024	NIROSHINI S	5	15	12	10	**42**
24	U22AE025	OBADAH AKRAM A	8	10	13	10	**41**
25	U22AE026	PANUGANTI SAI KRISHNA	7	15	10	10	**42**
26	U22AE027	PENNADA PRUDHVI NAGARJUNA	10	15	15	10	**50**
27	U22AE028	PINDIPOLU SANJAY	10	15	15	10	**50**
28	U22AE029	N RAJASEKHAR REDDY	10	15	15	10	**50**
29	U22AE030	REVANTH BOTUKA	6	8	8	10	**32**
30	U22AE031	SANJAY P	8	10	11	10	**39**
31	U22AE032	SHAIK ABDUL RAZAK	8	10	12	10	**40**
32	U22AE033	SHANMUGAPRIYA S	10	13	13	10	**46**
33	U22AE034	SHINODH G KURUP	7	12	11	10	**40**
34	U22AE035	SIVA RAJA PANDIAN S	8	12	11	10	**41**
35	U22AE036	SUSEENDAR R	8	13	13	10	**44**
36	U22AE037	SWARNIM KUMAR KAIWARTYA	10	15	15	10	**50**
37	U22AE038	TAVVA SIVANADHA REDDY	8	11	12	10	**41**

38	U22AE039	TIRUMALARAJU VENKATA NARASIMHA RAJU	6	11	10	10	**37**
39	U22AE040	UTLA GNANA DURGA MAHESH RAJU	10	15	15	10	**50**
40	U22AE041	VENU RACHAKONDA	7	11	11	10	**39**
41	U22AE042	YARRA HEMANTH	7	12	9	10	**38**
42	U22AS001	ADISHWAR R S P	7	11	10	10	**38**
43	U22AS002	AROKYA NISHA S	10	15	15	10	**50**
44	U22AS003	ARUNKUMAR D	9	11	10	10	**40**
45	U22AS004	BALAJI R	7	13	13	10	**43**
46	U22AS005	BAYAMUTHAKA DHANUNJAYA	8	15	12	10	**45**
47	U22AS006	CHINNAKARKALA SATISH REDDY	8	14	14	10	**46**
48	U22AS007	EROTHI MOHIT SAI SARVANANDH	10	15	15	10	**50**
49	U22AS008	HEIJINI B	10	15	15	10	**50**
50	U22AS009	JAYASURYA V	8	13	13	10	**44**
51	U22AS010	KADALI SHAHANYU	5	15	15	10	**45**
52	U22AS011	KARTHIK Y	10	15	15	10	**50**
53	U22AS012	KHUSHI BHATTA	10	15	15	10	**50**
54	U22AS013	KHUSHI SINGH	8	14	14	10	**46**
55	U22AS014	MOHAMED ZAIM M	10	15	15	10	**50**
56	U22AS015	MOHAMMED IBRAHIM A	9	15	14	10	**48**
57	U22AS016	MONIKA M	10	15	15	10	**50**
58	U22AS017	MURUGANANDAN GOKHUL SHANKAR	8	13	15	10	**46**
59	U22AS018	NIKHIL V	8	13	14	10	**45**
60	U22AS019	POTHIRAJA M	10	15	15	10	**50**
61	U22AS020	PRIYADARSHAN S	8	15	15	10	**48**
62	U22AS021	ROWENA MARY A	0	0	0	0	**0**
63	U22AS022	K SANJAY KUMAR REDDY	10	15	15	10	**50**
64	U22AS023	SARAN R	6	11	6	10	**33**
65	U22AS024	SAURABH KUMAR	8	13	14	10	**45**
66	U22AS025	SHIVA B	10	15	15	10	**50**
67	U22AS026	SNEHA J L	8	12	12	10	**42**
68	U22AS027	T S SRI SURYA TEJA	9	15	15	10	**49**
69	U22AS028	THOLUCHURI MANIKANTA	7	15	13	10	**45**
70	U22AS029	TORLAPATI ABHILASH	9	15	13	10	**47**
71	U22AS030	VARUN ESWARAN B	6	11	12	10	**39**
72	U22AS031	VEDAVARSHAA P	9	15	15	10	**49**
73	U22AS032	VIGNESH G	10	15	11	10	**46**
74	U22AS033	VISHALI J	9	15	15	10	**49**
75	U22AS034	YOGENDIRAN R	10	15	15	10	**50**
76	U22AS035	PRIYADARSHINI A	8	13	15	10	**46**

COURSE FILE 2022 - 2023 TERM – II ACCA / U20MABT02

STAFF NAME: Dr. R. ANBU

DEPARTMENT OF MATHEMATICS

DA \ HR	I 9.00-9.50	II 9.50-10.40		III 10.50-11.40	IV 11.40-12.30		V 1.30-2.20	VI 2.20-3.10	VII 3.10-4.00
MON			B		ACCA SEC B1	B			
TUES	ACCA SEC B1		R			R			
WED			E			E	ACCA SEC B1		
THUR			A			A			ACCA SEC B1
FRI			C K			C K		ACCA SEC B1	

BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT TEST-III
U20MABT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS

SET- A

Date & Session	: 31.01.2023 & FN
Academic Year / Term	: 2022-2023 / I
Duration	: 1 Hour 30 Minutes
Maximum marks	: 30 Marks
Instructions	: Answer to ALL questions

Q.No.	PART – A (5 x 2 = 10) Answer to All Questions	Weightage	CO's	Bloom's Level
1	Find the fixed points of the transformation $w=\frac{6z-9}{z}$	2	CO4	2
2	Write the Cauchy's integral formula for Derivatives.	2	CO5	1
3	What is the singular point of the function $f(z) = \frac{1}{z+i}$.	2	CO5	1
4	Evaluate $f(z)=\frac{z}{(z+1)(z+2)}$ about z = - 2.	2	CO5	3
5	Find the residue at z = 0 for $f(z)=\cot z$	2	CO5	3
	PART – B (2 x 4 = 8) Answer to All Questions			
6	(a) Determine the Analytic function f(z), whose real part is $x^3-3xy^2+3x^2-3y^2+1$ **(OR)** (b) Find the bilinear transformation that maps the points $z=0,-i,-1$ onto the points $w=i,1,0$ respectively.	4	CO4	2
7	(a). Evaluate $f(z)=\cos z$ as a Taylor's series about the point $z=0$. **(OR)** (b) Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}dz$, where C is the circle $\|z\|=1$	4	CO5	3
	PART – C (1 x 12 = 12) Answer to All Questions			
8	(a) Find f(z), if the Imaginary part is $e^x(x\sin y+y\cos y)$. Also find its conjugate. **(OR)** (b) Find the image of $\|z-2i\|=2$ under the transformation w=1/z	12	CO4	2

CO's	Weightage
CO 4	18
CO 5	12
Total	**30**

TERM - I

U20MABT02 : ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA - III.

Part - A

1. Find the fixed pts of the transformation $w = \frac{6z-9}{z}$

$z = 3, 3$

2. Write the Cauchy's integral formula for Derivatives

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} \, dz$$

$$f'(z_0) = \frac{1!}{2\pi i} \int \frac{f(z)}{(z-z_0)^2} \, dz$$

$\vdots \qquad \vdots$

3. What is the singular pt of the function $f(z) = \frac{1}{z+i}$

$z + i = 0$

$z = -i$

4. Evaluate $f(z) = \frac{z}{(z+1)(z+2)}$ about $z = -2$.

$$= \frac{\frac{z}{z+1}}{z+2}$$

$z + 2 = 0$, $z = -2$

$$\int \frac{f(z)}{z - z_0} = 2\pi i \, f(-2)$$

$$= 4\pi i$$

5. Find the residue at $z = 0$ for $f(z) = \cot z$.

$$f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)} = \frac{\phi(z)}{\psi'(z)} = \frac{\cos(z)}{\cos(z)}$$

$z = 0$, $\frac{\cos(0)}{\cos(0)} = 1$.

Part - B

6 (a) Determine the Analytic function $f(z)$, whose real part is

$x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ — ①

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ | $\frac{\partial u}{\partial y} = 0 - 6xy + 0 - 6y + 0$

$\phi_1(z,0) = 3z^2 + 6z$ | $\phi_1(z,0) = 0.$

By using Milne thomson Method.

$$f(z) = \int \{ \phi_1(z,0) - i\,\phi_2(z,0) \} dz$$

$$= \frac{3z^3}{3} + \frac{6z^2}{2}$$

$$f(z) = z^3 + 3z^2.$$

(B) Find the bilinear transformation that that maps the pts $z = 0, -i, -1$ onto the pts $w = i, 1, 0$ respectively

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{w-i}{wi+w} = \frac{-zi+z}{-(i+zi)}$$

$$w = \frac{z+1}{i-zi}$$

7 (A) Evaluate $f(z) = \cos z$ as a Taylor's series about the pt $z = 0$.

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$f(z) = \cos z$

$f(z) = \cos z$, $f(0) = \cos(0) = 1$

$f'(z) = -\sin z$ $\quad f'(0) = -\sin(0) = 0$

$f''(z) = -\cos z$ $\quad f''(0) = -1$

$$f(z) = 1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots$$

2

B) Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}\,dz$, where C is the circle $|z|=1$.

$z^2+4=0$ | $z-2=0$

$z^2=-4$ | $z=2$.

$z=\pm 2i$

$z=\pm 2i, 2$ are the singular pts

$z=\pm 2i$, and 2 are outside the ~~circle~~ circle $|z|=1$.

$\therefore \int_C f(z) = 0.$

Part-C

8 (A) Find $f(z)$, if the imaginary part is $e^x(x\sin y+y\cos y)$. Also find its conjugate.

$V = e^x(x\sin y + y\cos y)$

$\frac{\partial v}{\partial y} = e^x . x\cos y + e^x[-y\sin y + \cos y]$

$\phi_1(z,0) = e^z . z + e^z(0+1) = (z+1)e^z$

$\frac{\partial v}{\partial x} = \sin y\,[e^x . 1 + x . e^x] + e^x y\cos y$

let $\phi_2(z,0) = 0+0=0.$

$f(z) = \int \{\phi_1(z,0) + i\,\phi_2(z,0)\}\,dz$

$f(z) = ze^z$

To find its conjugate: $f(z) = ze^z$

$u = e^x(x\cos y - y\sin y)$

is the conjugate //

(8) Find the image of $|z-2i|=2$ under the transform $w=1/z$.

$$|z-2i|=2, \quad w=1/z$$

$z=x+iy$,

$f(z)=u+iv$

$$u+iv=\frac{1}{x+iy}$$

$$x+iy=\frac{1}{u+iv}\times\frac{u+iv}{u+iv}$$

$$x=\frac{u}{u^2+v^2}, \quad y=\frac{-v}{u^2+v^2}$$

$$|z-2i|=2$$

$$|x+iy-2i|=2$$

$$|x+i(y-2)|=2$$

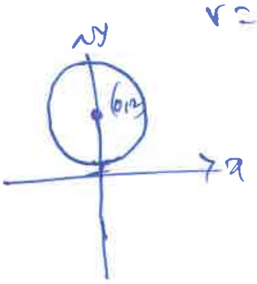
$$(x-0)^2+(y-2)^2=4$$

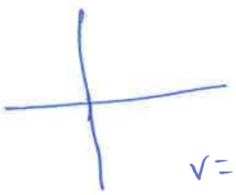
$$x^2+y^2-4y+4=4$$

$$x^2+y^2-4y=0$$

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$

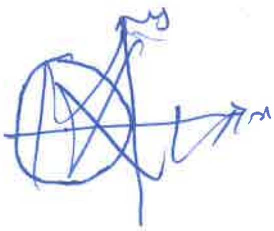
$$v=\frac{-1}{4}$$





$v=\frac{-1}{4}=0$

$\cancel{x}= -\frac{1}{4}$



BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY
Department of Mathematics

CONTINUOUS LEARNING ASSESSMENT TEST-II
U20MABT02 – ADVANCED CALCULUS AND COMPLEX ANALYSIS
SET - B

Date : 23.12.2022
Academic Year / Semester : 2022 – 2023 / ODD
Maximum marks : 30 Marks
Instruction : Answer to ALL questions

Q. No.	PART – A (5 x 2 = 10)	Weightage	CO's	Bloom's Level
1	State Gauss Divergence Theorem.	2	CO2	1
2	Prove that $L[Sinh\ at] = \frac{a}{s^2-a^2}$	2	CO3	2
3	Find $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$	2	CO3	2
4	Find $L[Sin\ 5t\ Cos\ 3t]$	2	CO3	2
5	Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$	2	CO3	2
	PART – B (2 x 4 = 8)			
6	(a) Find $L[e^{3t}Sin\ 2t]$ **(OR)** (b) Find $L[\ t^2\ Cosh\ 3t]$	4	CO3	3
7	(a) Using convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right]$. **(OR)** (b) Using partial fraction method to find $L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right]$.	4	CO3	3
	PART – C (1 x 12 = 12)			
8	(a) Verify Green's theorem in a plane for the integral $\int_C (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **(OR)** (b) Verify Gauss Divergence Theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the surface of the rectangular parallelepiped bounded by $x = 0$, $x = a,\ y = 0, y = b,\ z = 0, z = c$.	12	CO2	3

CO's	Weightage
CO 2	14
CO 3	16
Total	**30**

TERM - I

U20MABT02 - ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA - II

Part - A

1. State Gauss Divergence theorem:

If $\vec{F}$ is a vector pt function,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv \quad [\because dv = dx\,dy\,dz]$$

2. Prove that $L[\sinh at] = \frac{a}{s^2-a^2}$

$$\sinh\theta = \frac{e^{\theta}-e^{-i\theta}}{2}$$

$$= \frac{1}{2}\left[e^{at} - e^{-at}\right]$$

$$= \frac{a}{s^2-a^2}$$

3. Find $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$

$$\Rightarrow aL[\sqrt{t}] + bL[t^{-1/2}] + L[c]$$

$$\Rightarrow \frac{a\sqrt{\pi}}{2s^{3/2}} + \frac{b\sqrt{\pi}}{s^{3/2}} + c/s$$

4. Find $L[\sin 5t \cos 3t]$

$$= \frac{8}{s^2+64} + \frac{2}{s^2+4}.$$

5. Find $L^{-1}\left[\frac{3s+2}{s^2-4}\right]$

$$= 3\cosh 2t + \sinh 2t$$

Part - B

6. (A) Find $L[e^{3t}\sin 2t]$

$$\sin at = \frac{a}{s^2+a^2} = \frac{2}{s^2+4}$$

$$e^{3t}\sin 2t = \frac{2}{(s-3)^2+4}$$

$$= \frac{2}{s^2-6s+13}$$

(B) Find $L[t^2 \cosh 3t]$

$$L[\cosh 3t] = \frac{s}{s^2-9} = F(s)$$

$$L[t^2 \cosh 3t] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2-9}\right]$$

$$= \frac{2s^3+54s}{(s^2-9)^3}$$

(A) Using Convolution theorem, find $L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right]$

$$L^{-1}\left[\frac{s^2}{(s^2+5^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+5^2}\right] * L^{-1}\left[\frac{s}{s^2+5^2}\right]$$

$$= \cos 5t * \cos 5t$$

$$= \frac{1}{2}\left[t \cos 5t + \frac{2}{10} \sin 5t\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right] = \frac{1}{2}\left[t \cos 5t + \frac{1}{5} \sin 5t\right]$$

(B) Using Partial fraction method to find $L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right]$.

Consider. $\frac{s+2}{s(s+4)(s-9)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-9}$ —— ①

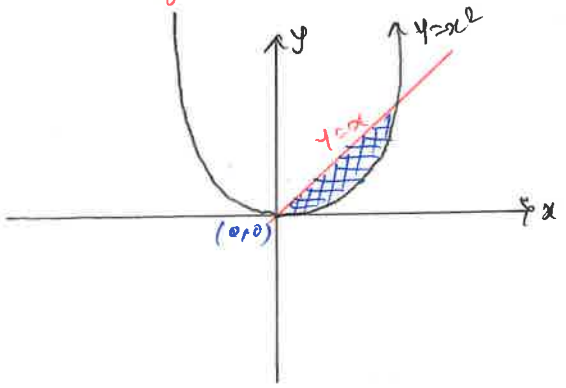
$A = -1/18$, $B = -1/26$, $C = 11/117$

$$L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right] = L^{-1}\left[\frac{-1/18}{s} + \frac{-1/26}{s+4} + \frac{11/117}{s-9}\right]$$

$$= \frac{1}{18} L^{-1}[1/s] - 1/26\, L^{-1}[1/(s+4)] + 11/117\, L^{-1}\left[\frac{1}{s-9}\right]$$

$$= -\frac{1}{18} - \frac{1}{26} e^{-4t} + \frac{11}{117} e^{9t}$$

8 (A) Verify Green's theorem $\int_C (xy+y^2)\,dx + x^2 dy$, where C is the closed curve of the region bounded by $y=x$ and $y=x^2$. ②



Green's theorem,

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.$$

$u = xy + y^2$ $\quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = x + 2y$

$v = x^2$

R.H.S

$$\iint_R (2x - x - 2y)\,dx\,dy = -\frac{1}{20}.$$

L.H.S

$$\int u\,dx + v\,dy$$

(i) Along OA $\left(y = x^2, \ \frac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx\right)$

$$\int_0^1 (x x^2 + (x^2)^2)\,dx + x^2(2x)\,dx = 19/20 \quad ①$$

(ii) Along AO ($y=x$ $\quad dy = dx$)

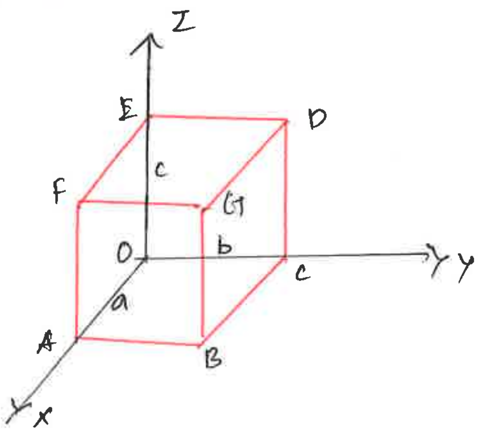
$$\int_1^0 (x x + x^2\,dx + x^2\,dx) = -1$$

Now adding ① 4 ②

$$\frac{19}{20} - 1 = \frac{19-20}{20} = -1/20$$

$\therefore$ L.H.S = R.H.S

(B) Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ and S is the surface of the rectangular parallelepiped bounded by $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.



Given

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2\ zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\iint_S \vec{F}\cdot d\hat{n}\,d\vec{s} = \iiint_V \nabla\cdot\vec{F}\ .dv$$

R.H.S

$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \vec{F}$

$= 2x + 2y + 2z$

$\iiint \nabla \cdot \vec{F} \cdot dv = \iiint (2x+2y+2z)\, dx\, dy\, dz$

$= abc\,(a+b+c)$

L.H.S

1) Along ABFG :

$a^2bc - \frac{b^2c^2}{4}$

(ii) Along OCDE :

$\frac{b^2c^2}{4}$

(iii) Along BCDG

$ab^2c - \frac{a^2c^2}{4}$

(iv) OAFE :

$\frac{a^2c^2}{4}$

(V) Along DEFG

$abc^2 - \frac{a^2b^2}{4}$

(vi) Along ABCO

$\frac{a^2b^2}{4}$

$\therefore \iint_S \vec{F} \cdot \hat{n}\, d\vec{s} = abc\,(a+b+c)$

$\therefore$ L.H.S = R.H.S

Gauss Divergence theorem proved.

radius 6 unit under the transformation $w = 3z$

(iv) $|z| = 1$, $w = 4z$

$w = f(z) = u + iv$

$u + iv = 4(x + iy)$

$u + iv = 4x + i4y$

$u = 4x$ | $v = 4y$

$\boxed{x = \frac{u}{4}}$ | $\boxed{y = \frac{v}{4}}$

$|z| = 1$

$|x + iy| = 1$

$\sqrt{x^2 + y^2} = 1$

$x^2 + y^2 = 1$

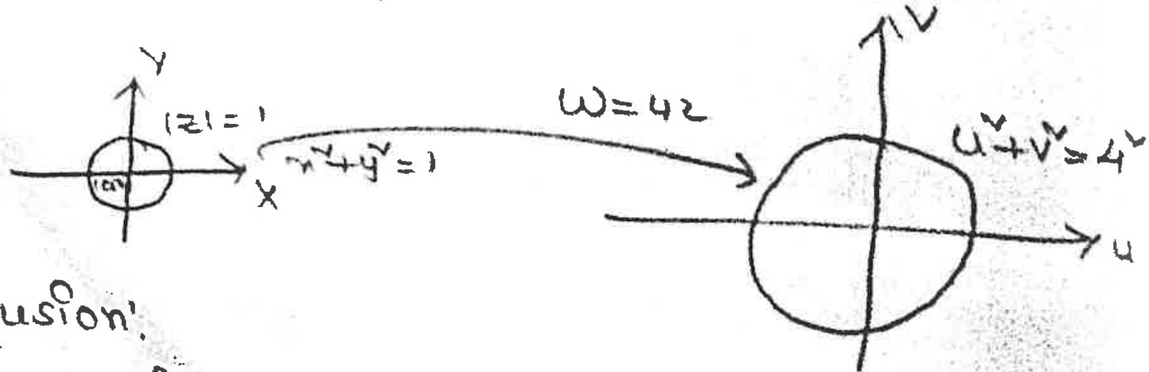
$\frac{u^2}{16} + \frac{v^2}{16} = 1$

$u^2 + v^2 = 16$

$\boxed{u^2 + v^2 = 4^2}$

Centre $(0,0)$

Radius $= 4$



Conclusion:

A Circle $|z| = 1$ with the Centre $(0,0)$ and radius 1 unit in a z-plane is transformed into a circle in w-plane with Centre $(0,0)$ and radius 4 unit under the transformation

(Q) Find Image of Circle $|z-2i|=2$ under the transformation $w=\frac{1}{z}$

Sol:- $|z-2i|=2$, $w=\frac{1}{z}$

$w=\frac{1}{z} \rightarrow ①$

$z=x+iy$

$w=f(z)$

$f(z)=u+iv$

$① \rightarrow u+iv=\frac{1}{x+iy}$

$x+iy=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$

$x+iy=\frac{u-iv}{u^2+v^2}$

$x+iy=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$

$x+iy=\frac{u}{u^2+v^2}+i\frac{(-v)}{u^2+v^2}$

$x=\frac{u}{u^2+v^2}$, $y=\frac{-v}{u^2+v^2}$

$|z-2i|=2$

$|x+iy-2i|=2$

$|x+i(y-2)|=2$

$x^2+(y-2)^2=4$

$(x-0)^2+(y-2)^2=4 \rightarrow$ circle

Centre $(0,2)$, Radius $=2$

$$(x-0)^2+(y-2)^2 = 2^2 = r^2$$

$$x^2+y^2-4y+4 = 4$$

$$\boxed{x^2+y^2-4y = 0}$$

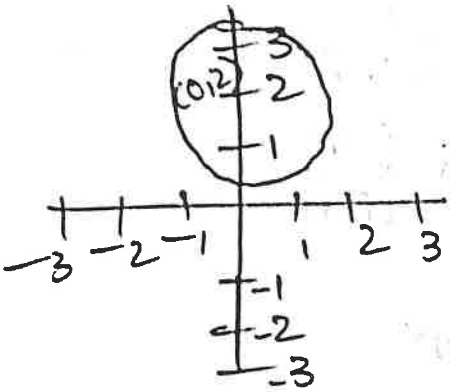
$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

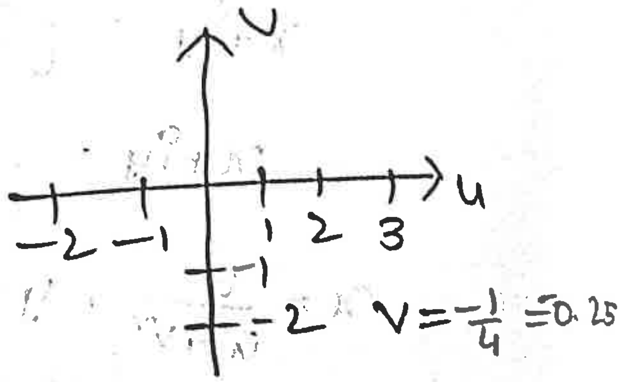
$$\frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}} + \frac{4v}{u^2+v^2}$$

$$1+4v = 0$$

$$4v = -1$$

$$\boxed{v = -\frac{1}{4}}$$





Conclusion:

An image of the circle $|z-2i| = 2$ with the centre (0,2) and radius 2 unit in a z-plane is transformed into a circle in ω-plane with centre (0,0) and radius 2 unit under transformation $\omega = \frac{1}{z}$

(iv) $|z+1|=1$, $\omega=\frac{1}{z}$

$$\omega=\frac{1}{z} \rightarrow (1)$$

$z=x+iy$, $\omega=f(z)$, $f(z)=u+iv$

By the previous problem

$$x=\frac{u}{u^2+v^2}, \quad y=\frac{-v}{u^2+v^2}$$

$$|z+1|=1 \Rightarrow |x+iy+1|=1$$

$$(x+1)^2+y^2=1^2$$

$$[x-(-1)]^2+(y-0)^2=1 \rightarrow \text{circle}$$

Centre $(-1,0)$ radius $=1$

$$(x+1)^2+y^2=1 \Rightarrow x^2+2x+1+y^2=1$$

$$x^2+2x+y^2=0$$

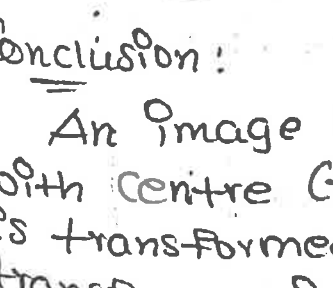
$$\left(\frac{u}{u^2+v^2}\right)^2+\frac{2u}{u^2+v^2}+\left(\frac{(-v)}{u^2+v^2}\right)^2=0$$

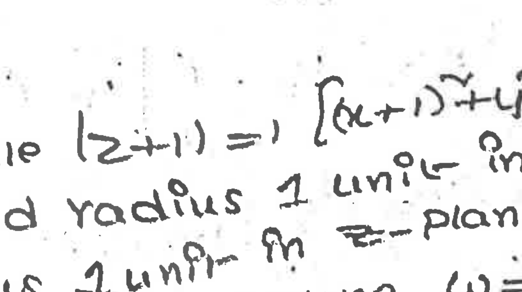
$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+\frac{2u}{u^2+v^2}$$

$$\frac{1+2u}{u^2+v^2}=0 \Rightarrow 1+2u=0$$

$$2u=-1$$

$$\boxed{u=\frac{-1}{2}}$$





Conclusion:

An image of circle $|z+1|=1$ $[(x+1)^2+y^2=1]$ in z-plane with centre $(-1,0)$ and radius 1 unit in z-plane is transformed into a $u=\frac{-1}{2}$ in ω-plane $\omega=\frac{1}{z}$

is transformed radius 1 unit in

Consuction of Analytic funtion by Milne thomson method

$F(z) = u + iv$

$f(z) = u(x,y) + iv(x,y)$

① $u(x,y)$ given

$x, y \rightarrow$ Independent Variable

$u \rightarrow$ dependent Variable

u_x, u_y

$\frac{du}{dx}$ & $\frac{du}{dy}$ exists

Assume:

$f(z) = u + iv$

$$f(z) = \underbrace{u(x,y)}_{\text{known}} + i\underbrace{v(x,y)}_{\text{unknown}}$$

Case-i $f'(z) = u_x(x,y) + i v_x(x,y)$

(Diff w.r.to x only)

$$f'(z) = u_x(x,y) + i v_x(x,y)$$

CR eqn $\begin{cases} u_x = v_y \\ u_y = -v_x \Rightarrow v_x = -u_y \end{cases}$

$$\boxed{f'(z) = u_x(x,y) - i u_y(x,y)}$$

$$\int f'(z)\,dz = \int [u_x(x,y) - i u_y(x,y)]\,dz$$

$$\int \frac{d}{dz}[f(z)]\,dz = \int [u_x(z,0) - i u_y(z,0)]\,dz$$

$$\boxed{f(z) = \int [u_x(z,0) - i u_y(z,0)]\,dz}$$

Case-ii Diff. w.r to 'y'

$$f'(z) = u_y(x,y) + i v_y(x,y)$$

$$f'(z) = u_y(x,y) + i u_x(x,y)$$

$$\boxed{f(z) = \int [u_y(z,0) + i u_x(z,0)]\, dz}$$

② $v(x,y)$ given

$x,y \rightarrow$ Independent Variable

$u \rightarrow$ dependent Variable

u_x, v_y

$\frac{du}{dx}$ & $\frac{dv}{dy}$ exists

Assume:

$$f(z) = u + iv$$

$$f(z) = \underbrace{u(x,y)}_{\text{unknown}} + i\underbrace{v(x,y)}_{\text{known}}$$

$$f'(z) = u_x(x,y) + i v_x(x,y)$$

Case-i

(Diff w.r to x only)

$$f'(z) = u_x(x,y) + i v_x(x,y)$$

CR eqn $\begin{cases} u_x = v_y \\ u_y = -v_x \Rightarrow v_x = -u_y \end{cases}$

$$\boxed{f'(z) = v_{xy}(x,y) + i v_x(x,y)}$$

$$\int f'(z)\, dz = \int [v_{xy}(x,y) + i v_x(x,y)]\, dz$$

$$\int \frac{d}{dz}[f(z)]dz = \int [V_y(z,0) + V_x(z,0)]dz$$

$$\boxed{f(z) = \int [V_y(z,0) + V_x(z,0)]dz}$$

Case-ii

Diff w.r to 'y'

$$f'(z) = u_y(x,y) + i v_y(x,y)$$

$$f'(z) = u_y(x,y) + i u_x(x,y)$$

$$\boxed{f(z) = \int [-V_x(z,0) + i V_y(z,0)]dz}$$

Problems:

① Construct the analytic function with f(z) whose real part is $e^x \cos y$.

Sol:- u_x, u_y exists

$$\frac{du}{dx} = e^x \cos y \Rightarrow u_x(z,0) = e^z \cos 0 = e^z$$

$$\frac{du}{dy} = e^x(-\sin y) = -e^x \sin y \Rightarrow u_y(z,0) = -e^z \sin 0$$
$$= 0$$

CR eqn $\begin{cases} u_x = v_y \\ v_x = -u_y \Rightarrow u_y = -v_x \end{cases}$

$$f'(z) = u_x + i v_x$$

$$f(z) = \int [u_x - i u_y] dz$$

$$f(z) = \int [u_x(z,0) - i u_y(z,0)]\,dz$$

$$f(z) = \int [e^z - i(0)]\,dz$$

$$f(z) = \int e^z\,dz$$

$$\boxed{f(z) = e^z}$$

put $z = x+iy$

$$f(z) = e^{x+iy} = e^x . e^{iy}$$

$$u+iv = e^x(\cos y + i\sin y)$$

$$u+iv = e^x\cos y + ie^x\sin y$$

$$\boxed{\begin{array}{l} u = e^x\cos y \\ v = e^x\sin y \end{array}}$$

Harmonic Function:

A function $u(x,y)$ is said to be harmonic if the 1st order partial derivative exists and continuous and satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (or) $u_{xx}+u_{yy}=0$

Similarly, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ (or) $v_{xx}+v_{yy}=0$

(*)

Show that $v = 3x^2y - y^3$

Sol:-

$$\frac{\partial v}{\partial x} = 6xy - 0 \qquad \frac{\partial v}{\partial y} = \cancel{}\ 3x^2 - 3y^2$$

$$\frac{\partial^2 v}{\partial x^2} = 6y \qquad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0$$

$\therefore$ $v(x,y)$ is harmonic

(*) Show that $u = 2x - x^3 + 3xy^2$ is harmonic

Sol:-

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \quad \Bigg| \quad \frac{\partial u}{\partial y} = 0 - 0 + \cancel{3x}\, 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \quad \Bigg| \quad \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

$\therefore u(x,y)$ is harmonic.

H.W

C.R eqn

$f(z) = \sin z$

$f(z) = \cos z$

**

(*) Construct the Analytic Function $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

Sol:-

$$u_x = \frac{\cosh 2y + \cos 2x\,(\cos 2x\,(2)) - \sin 2x\,(-\sin 2x)(2)}{(\cosh 2y + \cos 2x)^2}$$

$$u_x \underset{\substack{x = z \\ y = 0}}{(x,y)} = \frac{(\cosh 2(0) + \cos 2z)\, 2\cos 2z + 2\sin^2 2z}{(\cosh 2(0) + \cos 2z)^2}$$

$$= \frac{(1 + \cos 2z)\, 2\cos 2z + 2\sin^2 2z}{(1 + \cos 2z)^2}$$

$$= \frac{2\cos 2z + (2\cos^2 2z + 2\sin^2 2z)}{(1 + \cos 2z)^2}$$

$$= \frac{2\cos 2z + 2(1)}{(1 + \cos 2z)^2}$$

$$= \frac{2(\cancel{1+\cos 2z})}{(1+\cos 2z)^2}$$

$$= \frac{2}{1+\cos 2z}$$

$$= \frac{\cancel{2}}{\cancel{2}\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

$$u_x(z,0) = \sec^2 z$$

$$u_y = \frac{\sin 2x\,(-1)}{(\cosh 2y + \cos 2x)^2}\left(\sinh 2y\,(2)\right)$$

$$\frac{d}{dy}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$$

$$u_y = \frac{-2\sin 2x \cdot \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

$$u_y(z,0) = \frac{-2\sin 2z\,\sinh 2(0)}{(\cosh 2(0) + \cos 2z)^2}$$

$$= 0$$

CR eqn $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$$f'(z) = u_x + iv_x$$

$$f(z) = \int (u_x - iu_y)\,dz$$

$$= \int [u_x(z,0) - iu_y(z,0)]\,dz$$

$$f(z) = \int [\sec^2 z - i(0)]\,dz$$

$$= \int \sec^2 z\,dz$$

$$\boxed{f(z) = \tan z}$$

(iv) $V = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$V_x = \frac{(\cosh 2y - \cos 2x)(\cos 2x)2 - \sin 2x(\sin 2x)(2)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y . \cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y . \cos 2x - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2(\cosh 2y . \cos 2x - 1)}{(\cosh 2y - \cos 2x)^2}$$

$$V_{x\,(z,0)} = \frac{2(\cosh 2(0) . \cos 2z - 1)}{(\cosh 2(0) - \cos 2z)^2}$$

$$= \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2}$$

$$= \frac{-2\cancel{(1 - \cos 2z)}}{(1 - \cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{1 - \cos 2z}$$

$$= \frac{-\cancel{2}}{\cancel{2}\sin^2 z} = \frac{-1}{\sin^2 z}$$

$$V_x = -\mathrm{cosec}^2 z$$

$$V_y = \frac{\sin x(-1)}{(\cosh 2y - \cos 2x)^2}(\sinh 2y)(2)$$

$$V_y(z,0) = 0$$

$f'(z) = \cancel{u_x} \cancel{V_x - iV_y}\ \cancel{u_x} + iv_y$

$\cancel{= -V_x + iV_y} = u_y + iv_y$

$f'(z) = -V_x + iV_y$

CR eqn

$u_x = V_y$

$u_y = -V_x$

u_y is Unknown Term

$$f(z) = \int[-V_x(z,0) + i\,V_y(z,0)]dz$$

$$= \int[\operatorname{cosec}^2 z + i(0)]dz$$

$$= \int \operatorname{cosec}^2 z\, dz$$

$$\boxed{f(z) = -\cot z}$$

(*) Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ also find Complex Conjugate

Sol:- $u_x = \frac{d}{dx}(u) = 3x^2 - 3y^2 + 6x - 0$

$u_x = 3x^2 - 3y^2 + 6x$

$u_x(z,0) = 3z^2 - 0 + 6z$

$u_x(z,0) = 3z^2 + 6z$

$u_y = 0 - 3x(2y) + 0 - 6y$

$= -6xy - 6y$

$\boxed{u_y(z,0) = 0}$

$f(z) = u_x + iv_y$

$f(z) = \int \left(u_x(z,0) - i\, u_y(z,0)\right) dz$

$= \int \left[(3z^2 + 6z) - i(0)\right] dz$

$= \int \cancel{3x}(3z^2 + 6z)\, dz$

$= \frac{\cancel{3}z^3}{\cancel{3}} + \frac{\overset{3}{\cancel{6}}z^2}{\cancel{2}}$

$\boxed{f(z) = z^3 + 3z^2}$

$z = x + iy$

$f(z) = (x+iy)^3 + 3(x+iy)^2$

$= \left[x^3 + 3x^2(iy) + 3x(iy)^2 + iy^3\right]$

$+ 3\left[x^2 + (iy)^2 + i2xy\right]$

$= \left[x^3 + i(3x^2y) - 3xy^2 - iy^3\right] + 3\left[x^2 - y^2 + i2xy\right]$

$= x^3 + i(3x^2y) - 3xy^2 - iy^3 + 3x^2 - 3y^2 + i6xy$

$= \underbrace{x^3 - 3xy^2 + 3x^2 - 3y^2}_{u(x,y)} + i\underbrace{(3x^2y - y^3 + 6xy)}_{v(x,y)}$

$\therefore$ $v(x,y)$ is complex conjugate of $u(x,y)$

$\Rightarrow v(x,y) = 3x^2y - y^3 + 6xy.$

(5.) Determine the analytic function whose imaginary part is $3x^2y - y^3 + 6xy$ also find complex conjugate 'u'.

Sol:-

$$V = 3x^2y - y^3 + 6xy$$

$$V_x = 6xy - 0 + 6y$$

$$V_x = 6xy + 6y$$

$$V_x(z,0) = 6z(0) + 0 = 0$$

$$V_y = 3x^2 - 3y^2 + 6x$$

$$V_y(z,0) = 3z^2 - 0 + 6z$$

$$= 3z^2 + 6z$$

$$f(z) = \int (V_y + iV_x)\, dz$$

$$= \int [3z^2 + 6z + i(0)]\, dz$$

$$= \frac{\cancel{3}z^3}{\cancel{3}} + \frac{\overset{3}{\cancel{6}}z^2}{\cancel{2}}$$

$$\boxed{f(z) = z^3 + 3z^2}$$

$z = x + iy$.

$$= x^3 - 3xy^2 + 3x^2 - 3y^2 + i(3x^2y - y^3 + 6xy)$$

$\therefore$ $u(x,y)$ is complex conjugate of $v(x,y)$

$$\Rightarrow u(x,y) = \cancel{3x}\; x^3 - 3xy^2 + 3x^2 - 3y^2$$

//

(*) Construct an analytic function $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$

$\frac{d}{dx} uv = u\frac{d}{dx}v + v\frac{d}{dx}u$

Sol:- $V(x,y) = e^x . x\sin y + e^x . y\cos y$

$V_x(x,y) = \sin y\left[e^x(1) + xe^x\right] + y\cos y . e^x$

$V_x(z,0) = 0$

$V_y(x,y) = e^x . x\cos y + e^x\left[y(-\sin y) + \cos y(1)\right]$

$V_y(z,0) = e^z . z\cos(0) + e^z\left[\cos(0)\right]$

$= ze^z(1) + e^z(1)$

$= ze^z + e^z$

$V_y(z,0) = e^z(z+1)$

$f(z) = \int (v_y + iv_x)\,dz$

$= \int (e^z(z+1) + i(0))\,dz$

$f(z) = \int (z+1)e^z\,dz$

$\boxed{\int u\,dv = uv - \int v\,du}$

$= \int ze^z dz + \int e^z dz$

$= \left[ze^z - (1)e^z\right] + e^z$

$= ze^z - \cancel{e^z} + \cancel{e^z}$

$\boxed{f(z) = ze^z}$

$z = x+iy$

$f(z) = (x+iy)e^{x+iy}$

$= (x+iy)e^{x}\cdot e^{iy}$

$= (x+iy)e^{x}(\cos y + i\sin y)$

$= (x+iy)(e^{x}\cos y + ie^{x}\sin y)$

$= xe^{x}\cos y + ixe^{x}\sin \cancel{x}y + iye^{x}\cos y + i^2 ye^{x}\sin y$

$= \underbrace{xe^{x}\cos y - ye^{x}\sin y}_{u(x,y)} + i\underbrace{[xe^{x}\sin x + ye^{x}\cos y]}_{V(x,y)}$

$u(x,y) = xe^{x}\cos y - ye^{x}\sin y$ is complex Conjugate

(*) Construct an analytic function $f(z)$ whose ~~imagi~~ real part is $u = xe^{x}\cos y - ye^{x}\sin y$

Sol:-

$u_x(x,y) = xe^{x}\cos y - ye^{x}\sin y$

$u_x(x,y) = \cos y\,(xe^{x} + e^{x}(1)) + y\sin y \cdot e^{x}$

$u_x(z,0) = \cancel{0}\ \cancel{x}\ ze^{z} + e^{z}$

$u_y(x,y) = xe^{x}(-\sin y) - e^{x}(y(\cos y) + \sin y(1))$

$\cancel{+}\ \cancel{-xe^{x}\sin y}$

$u_y(z,0) = ze^{z}(-\sin(0)) - e^{z}(0(\cos(0)) + \sin(0))$

$= 0$

$f(z) = \int (u_x(z,0) - u_y(z,0))\,dz$

$= \int (ze^{z} + e^{z} - i(0))\,dz$

$$\boxed{f(z) = \int z e^{z}}$$

$z = x + iy$

(x) If $f(z)$ is analytic function the prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$$

Sol:- Consider

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \rightarrow (1)$$

eq (1) multiply by $|f(z)|^p$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = 4 \frac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^p$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left(|f(z)|^2\right)^{p/2} \qquad |z|^2 = z\bar{z}$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left(|f(z) f(\bar{z})|\right)^{p/2}$$

$$= 4 . \frac{\partial^2}{\partial z \partial \bar{z}} (f(z))^{p/2} (f(\bar{z}))^{p/2}$$

$$= 4 . \frac{\partial}{\partial z} (f(z))^{p/2} \frac{\partial}{\partial \bar{z}} (f(\bar{z}))^{p/2}$$

$$= \cancel{4} . \frac{p}{\cancel{2}} (f(z))^{\frac{p}{2}-1} f'(z) \frac{p}{\cancel{2}} (f(\bar{z}))^{p/2-1} . f'(\bar{z})$$

$$= p^2 (f(z))^{\frac{p}{2}-1} (f(\bar{z}))^{\frac{p}{2}-1} f'(z) . f'(\bar{z})$$

$$= p^2 (f(z) \ldots$$

$$= p^2 \left(|f(z)|^2\right)^{\frac{p}{2}-1} |f'(z)|^2$$

$$= p^2 . |f(z)|^{p-2} . |f'(z)|^2 = R.H.S \quad //$$

Unit-5

Complex Integration

~~Taylor Series:~~

Taylor Series:

The Taylor Series expansion for

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a)$$

$$+ \frac{(z-a)^4}{4!} f^{iv}(a) + \ldots$$

(or)

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n \times f^n(a)}{n!}$$

Put $z=0$ $a=\infty$

The Taylor Series above at $z=0$ is called meclauren Series.

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \frac{z^4}{4!} f^{iv}(0) + \ldots$$

① $f(z) = \sin z$ at $z=0$, $z=\frac{\pi}{4}$

$f(z)=\sin z$	at $z=0$	$z=\frac{\pi}{4}$
$f(z) = \sin z$	$f(0) = \sin 0 = 0$	$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f'(z) = \cos z$	$f'(0) = \cos 0 = 1$	$f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f''(z) = -\sin z$	$f''(0) = -\sin 0 = 0$	$f''(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$
$f'''(z) = -\cos z$	$f'''(0) = -\cos 0 = -1$	$f'''(\frac{\pi}{4}) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$
$f^{iv}(z) = \sin z$	$f^{iv}(0) = \sin 0 = 0$	$f^{iv}(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f^{v}(z) = \cos z$	$f^{v}(0) = \cos 0 = 1$	$f^{v}(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f^{vi}(z) = -\sin z$	$f^{vi}(0) = -\sin 0 = 0$	$f^{iv}(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

The Taylers Series explanation at $z=0$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \frac{z^4}{4!} f^{iv}(0) + \ldots$$

$$\sin z = 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(0) + \frac{z^3}{3!}(-1) + \frac{z^4}{4!}(0) + \frac{z^5}{5!}(1) + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots$$

The Tailor Series expantion at $z = \frac{\pi}{4}$ is giver

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \ldots$$

$$= f(\tfrac{\pi}{4}) + \frac{(z-\pi/4)}{1!} f'(\tfrac{\pi}{4}) + \frac{(z-\pi/4)^2}{2!} f''(\tfrac{\pi}{4}) + \ldots$$

$$= \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1}\left(\frac{1}{\sqrt{2}}\right) + \frac{(z-\pi/4)^2}{2}\left(-\frac{1}{\sqrt{2}}\right) + \ldots$$

$$= \frac{1}{2}\left[1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2} + \frac{(z-\pi/4)^3}{6} + \ldots\right]$$

(*) $F(z) = \cos z$ at $z=0$, $z = \frac{\pi}{4}$

Sol:

$f(z) = \cos z$	at $z=0$	at $z = \pi/4$
$f(z) = \cos z$	$f(0) = \cos 0 = 1$	$f(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f'(z) = -\sin z$	$f'(0) = \sin 0 = 0$	$f'(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$
$f''(z) = -\cos z$	$f''(0) = -\cos 0 = -1$	$f''(\frac{\pi}{4}) = -\cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$
$f'''(z) = \sin z$	$f'''(0) = \sin z = 0$	$f'''(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
$f^{iv}(z) = \cos z$	$f^{iv}(0) = \cos 0 = 1$	$f^{iv}(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

at $z=0$

$$f(z) = f(z) + \frac{(z-a)}{1!}f'(z) + \frac{(z-a)^2}{2!}f''(z) + \frac{(z-a)^3}{3!}f'''(z)$$

$$= f(0) + \frac{(z)}{1!}f'(0) + \frac{(z)^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0)$$

$$\cancel{= 1 + \frac{z}{1}\left(-\frac{1}{\sqrt{2}}\right) + \frac{z^2}{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{z^3}{3!}\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \cancel{1 - \frac{z}{\sqrt{2}} +} = \cancel{1 - \frac{z}{\sqrt{2}} - \frac{z^2}{2\sqrt{2}} + \frac{z^3}{6\sqrt{2}}}$$

$\neq$

$$= 1 + \frac{z}{1}(0) + \frac{z^2}{2}(-1) + \frac{z^3}{3!}(0)$$

$$= 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots$$

Cauchy Integral Theorem:

If $f(z)$ is analytic

$f'(z)$ is continuous

then.

$$\int_C f(z)\,dz = 0$$

Cauchy Integral Formula:

If $f(z)$ is analytic inside and on a Simple closed Curve 'C'. Let a be a Singular point of $f(z)$ and if $f(z)$ is not analytic at $z=a$ and ~~f(z) is~~ then the Cauchy integral formula is given by

$$\int_C \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$$

Cauchy integral formula for derivatives:

If $f(z)$ is analytic inside and on a Simple Closed Curve 'C'. Let a be a Singular point of $f(z)$ then the Cauchy integral formula is given by

$$\int \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(a)$$

(or)

$$f^{n}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Put $n=0$

$$f^{(0)}(a) = f(a) = \frac{1}{2\pi i} \int \frac{f(z)\,dz}{(z-a)}$$

$n=1, \quad f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^2}$

$n=2,$

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-a)^{2+1}}$$

$n=3, \quad f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{3+1}}\,dz$

① Evaluate using Cauchy integral formula

$$\int \frac{(z+1)}{(z-1)(z-3)}\,dz \qquad c: |z|=2$$

Sol:- $\frac{1}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$

$$1 = A(z-3) + B(z-1)$$

at $z=1$ $\quad 1 = A(1-3)+0$ $\quad$ at $z=3$

$\quad 1 = B(2)$

$-2A = 1$

$\boxed{A = \frac{-1}{2}}$ $\qquad \boxed{B = \frac{1}{2}}$

$$\Rightarrow \int \frac{1}{(z-1)(z-3)} = -\frac{1}{2} \cdot \frac{1}{(z-1)} + \frac{1}{2}\frac{1}{(z-3)}$$

$$\Rightarrow \int \frac{(z+1)}{(z-1)(z-3)}\,dz = \int \frac{-1}{2} \frac{(z+1)}{(z-1)} + \frac{1}{2}\int \frac{(z+1)}{(z-3)}$$

$$= -2\pi i f(1)\left(\frac{-1}{2}\right) + 0$$

$a=1,\quad f(z) = (z+1)$

$f(1) = 1+1 = 2$

$= -\frac{1}{\cancel{2}}\left[\cancel{2}\pi i(2)\right]$

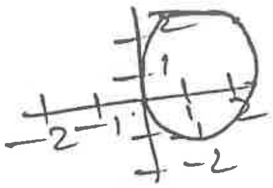
$= -2\pi i$

$|z-a| = r$

$|z| = 2$

$|z-0| = 2$

$a=0,\ r=2$



② $\int \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}\,dz,\quad |z|=3$

Sol: $\int \frac{f(z)}{(z-a)}\,f(z) = 2\pi i f(a) \rightarrow ①$

Consider

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} \rightarrow ②$$

$$1 = A(z-2)+B(z-1)$$

Put $z=1$

$1 = A(1-2)$

$\boxed{A=-1}$

put $z=2$

$1 = B(2-1)$

$\boxed{B=1}$

$$\Rightarrow \frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

③ $\Rightarrow \int_C \frac{\sin\pi z^2+\cos\pi z^2}{((z-1)(z-2)}\,dz$

$$= -\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}\,dz + \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}\,dz$$

Apply theorem

$$= -2\pi i f(1) + 2\pi i f(2)$$

$|z| = 3 \quad (z-1)(z-2) = 0$

$z-1=0, \quad z-2=0$

$z=1, z=2$ are Singular points

$\therefore f(z)$ is not analytic at $z = 1, 2$

$|z| = 3$

At $z=1$, $|1| = 3$

$1 < 3$

At $z=2$ $|2| = 3$

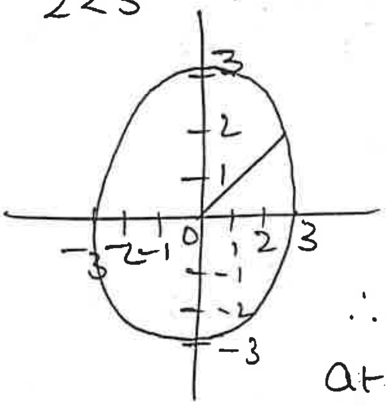
$2 < 3$

$|z| = 3$

$|z-0| = 3$

$|z-a| = r$

$a = 0, r = 3$



$\therefore f(z)$ is not defined at $z = 1, 2$ in $|z| = 3$ only

$\therefore f(z) = \sin\pi z^2 + \cos\pi z^2$

$f(1) = \sin\pi(1) + \cos\pi(1)$

$= 0 + (-1)$

$= -1$

$f(2) = \sin 4\pi + \cos 4\pi$

$= 0 + 1$

$f(2) = 1$

$$\Rightarrow \int \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}dz = -2\pi i(-1) + 2\pi i(1)$$

$= +2\pi i + 2\pi i$

$= 4\pi i$ //

③ $\int_C \frac{\cos\pi z^2}{(z-1)(z-2)}dz \quad |z| = 3$ (By Previous Problem)

Sol:-

$\therefore f(z) = \cos\pi z^2$

$f(1) = \cos\pi(1)^2$

$f(1) = -1$

$f(2) = \cos 4\pi$

$= 1$

$f(2) = 1$

$$\therefore \int \frac{\cos\pi z^2}{(z-1)(z-2)}dz = -2\pi i(-1) + 2\pi i(1)$$

$= 2\pi i + 2\pi i$

$= 4\pi i$ //

④ $\int \frac{4z^2 - 4z + 1}{(z^2+4)(z-2)}dz, \quad |z| = 1$

Sol:-

$$\int \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

Consider,

$(z^2+4)(z-2)=0$

$z^2=-4$

$z^2=(i)^2 4$

$z=\pm 2i$

$z=\sqrt{-4}$

$z=$

$z^2+4=0$	$z-2=0$
$z^2=-4$	$z=2$
$z^2=\sqrt{-4}$	
$z=\pm 2i$	

The Singular points are $z=\pm 2i, 2$

$z=+2i\ ,\ -2i\ ,\ 2$

$=0+2i\ ,\ 0-2i\ ,\ 2-i0$

$z=x+iy$ $\quad z=x+iy$

$z=0+i2$ $\quad z=2+i(0)$

points $=(0,2)\ ,(0,-2)\quad(2,0)$

$\sqrt{0^2+2^2}=2$

$\lvert z\rvert=1$	$\lvert -2i\rvert=1$	$\lvert 2+i0\rvert=1$
$\lvert 2i\rvert=1$	$0^2+(-2)^2=1$	$2^2=1$
$(0^2+2^2)=1$	$4>1$	$4>1$
$4>1$		

$\therefore$ All the points lies out Side $|z|=1$

$$\int_C f(z)\,dz=0$$ (Cauchy Fundamental theorem)

$$\therefore \int_C \frac{(4z^2-4z+1)}{(z^2+4)(z-2)}\,dz=0$$ //

(*) $\displaystyle\int \frac{1}{z+2}\,dz \qquad |z-1|=\frac{1}{2}$

Sol:-

$$\int \frac{f(z)}{z-a}\,dz=2\pi i f(a)$$

$f(z)=1\ \ \&\ \ a=-2$

Singular point $z = -2$

$\therefore \dfrac{f(z)}{z-a}$ is not analytic

at $z = a$

$\Rightarrow \dfrac{1}{z+2}$ is not analytic

at $z = -2$ $\left(\dfrac{1}{-2+2} = \dfrac{1}{0} = \infty\right)$

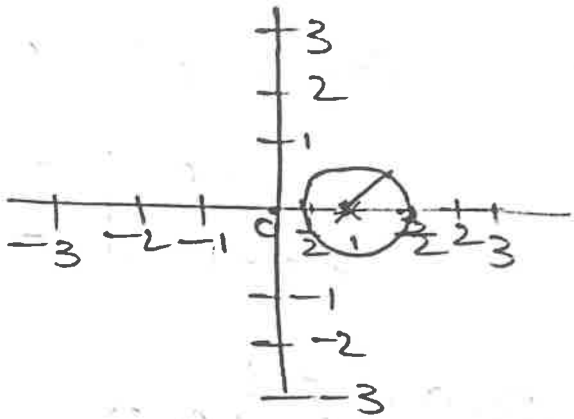
$|z-1| = \frac{1}{2}$

$|z-a| = r$

Centre 1, $r = 1/2$

Centre $1 + i0$

Point $(1, 0)$



-2 lies outside $|z-1| = \frac{1}{2}$

By Cauchy Fundamental theorem //

$|z-1| < \frac{1}{2}$

$1 - \frac{1}{2} < z < 1 + \frac{1}{2}$

$\frac{1}{2} < z < \frac{3}{2}$

Singularity:

Types of Singularity:

1. Removal Singularity
2. Poles
3. Essential Sigularity

Note:

1. $\lim_{z \to a} f(z) = $ finite ($\frac{0}{0}$ form / indeterminate case)
2. $\lim_{z \to a} f(z) = \infty \longrightarrow$ Pole $\longrightarrow$ Single pole & Multiple Pole

Ex: $\lim_{z \to 1} \frac{1}{z-1} = \infty$

③ $\lim_{z \to 1} \frac{1}{(z-1)^2} = \lim_{z \to 1} \frac{1}{(z-1)(z-2)} = \infty$

④ $\lim_{z \to a} f(z) =$ does not exists

Problems:

① $f(z) = \frac{z}{(z+1)(z-i)}$

Sol:- $(z+1)(z-i) = 0$

$z+1 = 0$ | $z-i=0$

$z=-1$ | $z=i$

$f(-1) = \frac{-1}{(-1+1)(-1-i)}$ $f(i) = \frac{i}{(i+1)(i-i)}$

$= \infty$

$= \infty$

$\therefore z=-1, i$ are Singular point

$z=-1, i$ Pole of order 1

② $f(z) = \frac{z}{(z+1)^2(z-i)^2}$

Sol:- $(z+1)^2(z-1)^2 = 0$

$(z+1)^2=0$ | $(z-i)^2=0$

$z+1=0$ | $z-i=0$

$z=-1,-1$ | $z=i,i$

$z = -1, -1, i, i$ is pole of order 2.

(f) $f(z) = \frac{1}{z+i} \left(= \frac{1}{0} = \infty \right)$

Sol: $z + i = 0$

$z = -i$ is simple pole.

(g) $f(z) = \frac{\sin z}{z}$

Sol: $z = 0$

$f(0) = \frac{\sin z}{z} = \frac{\sin 0}{0} = \frac{0}{0}$

$f(z) = \frac{\sin z}{z}$

$= \frac{\cos z}{1}$

At $z=0$

$f(0) = \frac{\cos z}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$ finite

$\therefore z=0$ is a removable singularity

$$f(z) = \frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z}$$

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

At $z=0$

$$f(0) = 1 - \frac{0}{3!} + \frac{0}{5!} - \frac{0}{7!} + \cdots$$

$= 1 - 0 = 1$

At $z=0$ is removable singularity

⑤ $f(z) = \frac{1}{z^2-2}$ $\left(= \frac{1}{0} = \infty\right)$

Sol:- $z^2 - 2 = 0$

$z^2 = 2$

$z = \pm\sqrt{2}$

$f(\sqrt{2}) = \frac{1}{(\sqrt{2})^2-2} = \frac{1}{0} = \infty$

$f(-\sqrt{2}) = \frac{1}{(-\sqrt{2})^2-2} = \frac{1}{2-2} = \frac{1}{0} = \infty$

$z = +\sqrt{2}, -\sqrt{2}$

is a simple pole.

⑥ $f(z) = \frac{1}{z(z-1)}$

$z(z-1) = 0$

$z = 0 \mid z = 1$ are simple pole.

⑦ $\underset{z \to a}{Lt} \frac{z^2-a^2}{z-a}$ $\left(= \frac{1}{0} = \infty\right)$

$z - a = 0 \mid z = a$

$f(z) = \frac{z^2-a^2}{z-a}$

At $z = a$ $f(a) = \frac{a^2-a^2}{a-a} = \frac{0}{0}$

$\underset{z \to a}{Lt} \frac{z^2-a^2}{z-a} = \underset{z \to a}{Lt} \frac{(z-a)(z+a)}{z-a}$

$= 2a$

$\therefore z = a$

Removable Singularity.

(8) Identify the nature of Singularity

$$f(z) = \frac{\sin z - z}{z^3}$$

Sol:- $z^3 = 0$

$z.z.z = 0$

$z = 0$ (3 times)

$f(0) = \frac{0}{0}$ form

$$f(z) = \frac{\sin z - z}{z^3}$$

$$g(z) = \frac{\cos z - 1}{3z^2}$$

$z=0$

$$g(0) = \frac{\cos 0 - 1}{3(0)^2} = \frac{0}{0}$$

$$g(z) = \frac{\cos z - 1}{3z^2}$$

$z=0$

$$h(z) = \frac{-\sin z}{6z}$$

$$h(0) = \frac{-\sin(0)}{6(0)} = \frac{0}{0}$$

$z=0$

$$k(z) = \frac{-\cos z}{6}$$

$$k(z) = \frac{-1}{6}$$

$\therefore \frac{\sin z - 1}{z^3}$ is Removable Singularity at $z=0$.

⑨ $f(z) = \dfrac{1-e^z}{z^4} \quad \left(= \dfrac{1}{0} = \infty\right)$

Sol:- $z^4 = 0$

$z = 0$ (4-times)

$f(0) = \dfrac{1-e^0}{(0)^4} = \dfrac{1-1}{0} = \dfrac{0}{0}$

$g(z) = \dfrac{0-e^z}{4z^3} = \dfrac{-e^z}{4z^3}$

$z=0$ $\quad g(0) = \dfrac{-e^0}{4(0)^3} = \dfrac{-1}{0} = \infty$

$z=0$ is a pole of order 3

for $g(z) = \dfrac{-e^z}{4z^3}$

⑩ $f(z) = e^{1/z}$

Sol:- At $z=0$, $f(0) = e^{1/0} = e^{\infty} =$ does not exists

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$f(z) = e^{1/z} = 1 + \frac{\frac{1}{z}}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots$$

$$= 1 + \frac{1}{z1!} + \frac{1}{z2!} + \frac{1}{z3!} + \cdots$$

$z=0$ $\quad = \underbrace{1 + \infty + \infty + \infty + \infty + \cdots}_{\text{infinite no. of times}}$

= Does not exists.

Residue's:

Residue's computations:

Simple pole:

Multiple pole / pole of order 'm'

Formula:

Residue of $f(z)$ at $z=a$ is a simple pole $\Big\} = \lim_{z \to a} (z-a) f(z)$

Multiple pole:

Residue of $f(z)$ at $z=a$ is a pole of order 'm' $\Big\} = \lim_{z \to a} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^n f(z) \right\}$

① Find the Residue of $f(z) = \frac{z^2}{(z-1)^2} = \left(\frac{1}{0} = \infty\right)$

Sol: $(z-1)^2 = 0$

$(z-1) = 0$ (two types)

$z=1$ is a pole of order 2

Residue of $f(z)$ at $z=a$ is a pole of order m $\Big\} = \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m \cdot f(z) \right\}_{z=a}$

$m=2, a=1$

$$= \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ \cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}} \right\}_{z=1}$$

$$= \frac{1}{1} \frac{d}{dz} \left\{ z^2 \right\}_{z=1}$$

$$= 2z = 2(1) = 2$$

(2) $f(z) = \dfrac{z}{(z+1)(z+2)}$

Sol:

$$(z+1)(z+2) = 0$$

$z+1=0$ | $z+2=0$

$z=-1$ | $z=-2$

$$f(-1) = \frac{-1}{(-1+1)(-1+2)} = \infty$$

$$f(-2) = \frac{-2}{(-2+1)(-2+2)} = \infty$$

$\therefore$ $z = -1, -2$ is a Simple pole

$\therefore$ Residue of $f(z)$ at $z=-1$ Simple pole $= \underset{z\to -1}{Lt} \left\{ (z-(-1)) \dfrac{z}{(z+1)(z+2)} \right\}$

$$= \underset{z\to -1}{Lt} \left[\cancel{(z+1)} \cdot \frac{z}{\cancel{(z+1)}(z+2)} \right]$$

$$= \frac{-1}{-1+2} = \frac{-1}{1} = -1$$

$z=-2$

Residue of $f(z)$ at $z=-2$ is a Simple pole $= \underset{z\to -2}{Lt} \left\{ [z-(-2)] \dfrac{z}{(z+1)(z+2)} \right\}$

$$= \underset{z\to -2}{Lt} \left\{ \cancel{(z+2)} \frac{z}{(z+1)\cancel{(z+2)}} \right\}$$

$$= \underset{z\to -2}{Lt} \frac{z}{z+1} = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

Q) Find the residue of $f(z) = \cot z$ at $z=0$

Sol:
$$f(z) = \frac{\sin z}{\cos z} \quad \frac{\cos z}{\sin z}$$

at $z=0$,
$$f(0) = \frac{\cos 0}{\sin 0} = \frac{1}{0} = \infty$$

$$f(z) = \frac{g(z)}{h'(z)}$$

Let $g(z) = \cos z$

$h(z) = \sin z$

$h'(z) = \cos z$

$$\left.\begin{array}{l}\text{Residue of } f(z) \\ \text{at } z=0\end{array}\right\} = \lim_{z \to 0} \frac{g(z)}{h(z)}$$

$$= \lim_{z \to 0} \frac{\cos z}{\cos z}$$

$$= \frac{\cos 0}{\cos 0} = \frac{1}{1} = 1$$

Cauchy's Residue Theorem:

Statement:

Let $f(z)$ is analytic inside and on a simple closed curve 'C' and $z_0, z_1, z_2, \ldots z_n$ be a finite no. of sigular point inside and on 'c' then the residue of $f(z)$ is given by

$$\int_C f(z)\,dz = 2\pi i \left[\begin{array}{l}\text{Sum of the residue of } f(x) \text{ at} \\ z = z_0, z_1, \ldots, z_n \text{ inside and} \\ \text{on C}\end{array}\right]$$

$$= 2\pi i \left[\begin{matrix} \text{Res. of } f(z) \\ \text{at } z = z_0 \end{matrix} + \begin{matrix} \text{Res of } f(z) \\ \text{at } z = z_1 \end{matrix} + \begin{matrix} \text{Res of } f(z) \\ \text{at } z = z_2 \end{matrix} \right]$$

(*) Determine the poles of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$. Find the residue value then calculate $\int \dfrac{z^2}{(z-1)^2(z+2)} dz$, $|z| = 3$.

Sol: To find poles

$$(z-1)^2(z+2) = 0$$

$$z-1=0 \quad | \quad z=-2$$

$$z=1 \quad | \quad z=-2$$

Poles are -2 (simple pole)

$z=1$ Pole of order 2

$$\left.\begin{matrix} \text{Residue of } f(z) \\ \text{at } z=-2 \text{ simpole} \end{matrix}\right\} = \lim_{z \to a} (z-a) f(z)$$

$$a = -2 \qquad = \lim_{z \to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 \cancel{(z+2)}}$$

$$= \lim_{z \to -2} \frac{z^2}{(z-1)^2}$$

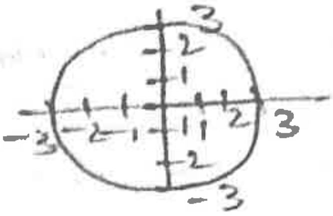
$$= \frac{4}{9}$$

$z=1$ (double pole)

$$\left.\begin{matrix} \text{Res. of } f(z) \text{ at} \\ z=1 \text{ double pole} \end{matrix}\right\} = \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right]$$

$$= \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{(z+2)}\right]_{z=1}$$



$$= \left[\frac{(z+2)(2z) - z^2(1)}{(z+2)^2}\right]_{z=1}$$

$$= \frac{6-1}{9} = \frac{5}{9}$$

$$\int_C f(z)\,dz = \int_{C,|z|=3} \frac{z^2}{(z-1)^2(z+2)}\,dz$$

$$= 2\pi i \left[\begin{matrix}\text{Res. of}\\ f(z) \text{ at}\\ z=1\end{matrix} + \begin{matrix}\text{Res. of}\\ f(z) \text{ at}\\ z=-2\end{matrix}\right]$$

$$= 2\pi i \left[\frac{4}{9} + \frac{5}{9}\right]$$

$$= 2\pi i \quad //$$

(*) If $|z| = \frac{1}{2}$ both Singular point lies out Side $\int_C f(z)\,dz = 0$

(*) Find the residue of $f(z) = \frac{z^2}{(z-1)^2}$ where 'c' is a circle $|z|=3$ (Applying Residue Theorem)

Sol:-

$$(z-1)^2 = 0$$

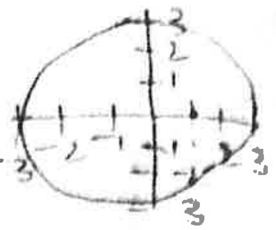
$$z-1 = 0$$

$$z = 1, 1 \quad \Big| \text{ Pole of order is } 2$$

Residue $|z|=3$

$|z-0|=3$

Centre $=0$, Radius $=3$



The pole $z=1$ lies inside $|z|=3$

Res. of $f(z)$ at $z=1$ is pole of order 2 $\Bigg\} = \dfrac{1}{(m-1)!}\left\{\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right\}_{z=a}$

$a=1,\ m=2$

$$= \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}(z-a)^2 f(z)\right\}_{z=1}$$

$$= \frac{1}{1!}\left\{\frac{d}{dz}\cancel{(z-1)^2}\frac{z^2}{\cancel{(z-1)^2}}\right\}_{z=1}$$

$$= \frac{d}{dz}\{(z)^2\}$$

$$= \{2z\}_{z=1} = 2(1) = 2$$

By Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i \left\{\begin{array}{l}\text{Sum of the}\\ \text{residue of } f(z) \text{ inside } c\end{array}\right\}$$

$$= 2\pi i \left[\begin{array}{l}\text{Res. of } f(z) \text{ at } z=1\\ \text{lies inside } |z|=3\end{array}\right]$$

$$= 2\pi i\,[2]$$

$$= 4\pi i$$

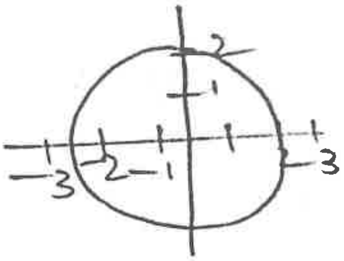
$$\int_C \frac{z^2}{(z-1)^2} dz = 4\pi i$$

Evaluate the Residue Value of $f(z) = \frac{z}{(z+1)(z+2)}$

$$\int_C \frac{z^2-3z+5}{z+4} dz, \ |z|=2$$

$$\int_C \frac{z^2-3z+5}{z+4} dz = \int_C f(z)dz$$

$$\therefore f(z) = \frac{z^2-3z+5}{z+4}$$



$z+4=0$

$z=-4$

$z=-4$ is a Simple pole

Singular Point $z=-4$ lies outside $|z|=2$

$\Rightarrow$ No need to complete residue value

By Cauchy Residue Theorem

$$\int_C f(z)dz = 2\pi i \left[\text{Sum of the residue of } f(z) \text{ inside } c \right.$$

$$= 2\pi i \left[\text{Res. of } f(z) \text{ at } z=-4 \text{ lies outside } |z|=2 \right]$$

$$= 2\pi i (0)$$

$$\therefore \int_C \frac{z^2-3z+5}{z+4} dz = 0$$

(*) $f(z) = \dfrac{1-e^{2z}}{z^3}$

$z^3 = 0 \quad z = 0$ (3 times)

$f(0) = \dfrac{1-e^{2(0)}}{0^3} = \dfrac{1-1}{0} = \dfrac{0}{0}$

$f(z) = \dfrac{1-e^{2z}}{z^3}$

$g(z) = \dfrac{0-e^{2z}(2)}{3z^2}$

$g(0) = \dfrac{0-2e^{2(0)}}{3(0)^2} = \dfrac{1}{0}$ (Pole of order 2)

$= \dfrac{-2e^{2z}}{3z^2}$

Pole of order 2

$\left.\begin{array}{l}\text{Res. of } f(z) \text{ at } z=a \\ \text{is pole of order } m\end{array}\right\} = \dfrac{1}{(m-1)!}\left[\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m g(z)\right]_{z=a}$

$a=0, m=2$

$= \dfrac{1}{1!}\left[\dfrac{d}{dz}(z-0)^2\left(\dfrac{-2e^{2z}}{3z^2}\right)\right]$

$= \dfrac{d}{dz}\left[\cancel{z^2}\dfrac{(-2e^{2z})}{3\cancel{z^2}}\right]$

$= \dfrac{-2}{3}\dfrac{d}{dz}(e^{2z})_{z=0}$

$= \dfrac{-2}{3}\left[(e^{2z}).(2)\right]_{z=0}$

$= \dfrac{-4}{3}e^{2(0)} = \dfrac{-4}{3}$

Laurentz's Series Expansion:

(1) Apply a Taylors Series expansion for $f(z)$

$f(z) = \frac{z^2-1}{(z+2)(z+3)}$ $|z| < 2$ & Laurentz Series

expansion $2 < |z| < 3$

Consider,

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{A}{z+2} + \frac{B}{(z+3)}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{(z+2)(z+3) + A(z+3) + B(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = (z+2)(z+3) + A(z+3) + B(z+2)$$

$z=-2$

$$4-1 = 0 + A(-2+3)$$

$$\boxed{A=3}$$

$z=-3$

$$9-1 = 0 + 0 + B(-3+2)$$

$\cancel{5B=8}$, $\cancel{B=\frac{8}{5}}$ | $B(-1) = 8$, $\boxed{B=-8}$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

$|z| < 2$, $\frac{|z|}{2} < 1$ | $\Rightarrow \frac{z^2+1}{(z+2)(z+3)} = 1 + \frac{3}{2\left(\frac{z}{2}+1\right)} - \frac{8}{3\left(\frac{z}{2}+1\right)}$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots\right] - \frac{8}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \right.$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$ is a Taylor Series.

By Laurentz's Series

$2 < \lvert z \rvert < 3$	$\lvert z \rvert < 3$
$2 < \lvert z \rvert$	$\frac{\lvert z \rvert}{3} < 1$
$\lvert z \rvert > 2$	
$\frac{2}{\lvert z \rvert} < 1$	

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \dots\right] - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^3 \dots\right.$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$ is a

$\Rightarrow$ Laurentz's Series.

Q) Find Laurent's Series expansion $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$

$1 < |z+1| < 3$

Sol: Given $1 < |z+1| < 3$

Let $u = z+1$ $\qquad$ $z-2 = u-1-2$

$u-1 = z$ $\qquad$ $= u-3$

$z+1 = u-1+1$

$= u$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{7(u-1)-2}{(u-1)(u-3)u}$$

$$= \frac{7u-9}{u(u-1)(u-3)} \rightarrow (1)$$

Consider, $\dfrac{7u-9}{u(u-1)(u-3)} = \dfrac{A}{u} + \dfrac{B}{u-1} + \dfrac{C}{u-3} \rightarrow (2)$

$$\frac{7u-9}{u(u-1)(u-3)} = \frac{A(u-1)(u-3) + B(u)(u-3) + C(u)(u-1)}{u(u-1)(u-3)}$$

$$7u-9 = A(u-1)(u-3) + B(u-3)u + C(u-1)u$$

$\underline{\underline{u=0}}$

$-9 = A(0-1)(0-3) + B(0) + C(0)$

$-9 = A(-1)(-3)$

$3A = -9$

$\boxed{A = -3}$

$\underline{\underline{u=1}}$

$7-9 = A(0) + B(1)(1-3) + C(0)$

$-2 = -2B$

$\boxed{B = 1}$

$\underline{u=3}$

$$12 = C(3-1)(3)$$

$$12 = C(2)(3)$$

$$12 = 6C$$

$$\boxed{C=2}$$

$$② \Rightarrow \frac{7u-9}{u(u-1)(u-3)} = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

$$1 < |z+1| < 3 \Rightarrow 1 < |u| < 3$$

$$u = z+1$$

$$1 < |u|, \quad |u| < 3$$

$$\frac{1}{|u|} < 1, \quad \frac{|u|}{3} < 1$$

$$\frac{7u-9}{u(u-1)(u-3)} = \frac{-3}{u} + \frac{1}{-1(1-u)} + \frac{2}{3\left(\frac{u}{3}-1\right)}$$

$$= \frac{-3}{u} - \frac{1}{(1-u)} - \frac{2}{3\left(1-\frac{u}{3}\right)}$$

$$= \frac{-3}{u} - 1(1-u)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{-3}{u} - \left(1+u+u^2+u^3+\dots\right) - \frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\dots\right]$$

$$= \frac{-3}{u} - \Sigma u^n - \frac{2}{3}\Sigma\left(\frac{u}{3}\right)^n$$

$$\therefore \frac{7z-2}{z(z-2)(z+1)} = \frac{-3}{z+1} - \Sigma(z+1)^n - \frac{2}{3}\left(\frac{z+1}{3}\right)^n$$

(*) Evoluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$ by Contour Integration

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$

$\cos\theta = \frac{z + \frac{1}{z}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$\sin\theta = \frac{1}{2i}\left(z - 1/z\right)$

$e^{i\theta} i\, d\theta = dz$

$d\theta = \frac{dz}{e^{i\theta} i}$

$d\theta = \frac{dz}{zi}$

Sol:- $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

$= \int_0^{2\pi} \frac{\frac{dz}{zi}}{5+3\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$

$= \int_0^{2\pi} \frac{\frac{dz}{zi}}{5+\frac{3z^2+3}{2z}}$

$= \int_0^{2\pi} \frac{dz}{\cancel{z}i} \times \frac{2\cancel{z}}{3z^2+10z+3}$

$= \frac{2}{i}\int_0^{2\pi} \frac{dz}{(z+3)\left(z+\frac{1}{3}\right)}$

Here, $z = -3, -\frac{1}{3}$ are Singular of $f(z)$

$z=-3$ is lies outside of Circle

$z = -\frac{1}{3}$ is lies inside of Circle $|z|=1$

$\left.\begin{array}{l}\text{Res. of } f(z) \\ \text{at } z=-\frac{1}{3} \text{ is} \\ \text{Simple pole}\end{array}\right\} = \lim_{z\to\frac{1}{3}} \cancel{\left(z+\frac{1}{3}\right)} \frac{1}{(z+3)\cancel{\left(z+\frac{1}{3}\right)}.3}$

$z \to -\frac{1}{3}$

$= \frac{1}{3\left(-\frac{1}{3}+3\right)} = \frac{1}{8}$

By Cauchy Res. theorem $= 2\pi i \left[\begin{matrix}\text{Sum of the residue}\\ \text{of } f(z)\end{matrix}\right]$

$$= \frac{2}{i}\left[2\pi i\left(\begin{matrix}\text{Res. of } f(z)\\ \text{at } z=-3\end{matrix}\right)\right]$$

$$= \frac{2}{i}\left[\frac{2\pi i}{8}\right] = \frac{4\pi \cancel{i}}{8\cancel{i}}$$

$$= \frac{4\pi}{8}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2} //$$

Type II Contour Integral:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \quad \text{deg. of } Q(x) > \text{deg. of } P(x)$$

We consider only positive poles

$$\therefore \int f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz$$

as $R \to \infty$ all positive poles lies inside upper semi-circle.

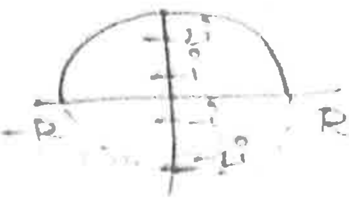
(*) $$\int_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2+1)(x^2+4)}$$

$$\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_{-R}^{R} \frac{x^2 dx}{(x^2+1)(x^2+4)} + \int_{\Gamma} \frac{z^2 dz}{(z^2+1)(z^2+4)}$$

→ (2)

$\int_C f(z)\,dz = 2\pi i$ [Sum of residues]

$f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}$



$(z^2+1)(z^2+4) = 0$

$z^2+1=0$	$z^2=-4$
$z^2=-1$	$z=\pm 2i$
$z=\pm i$	

$z = i, -i, 2i, -2i$

$z = i, 2i$ lies upper semi circle

$z = -i, -2i$ lies below x-axis

To find residue of $f(z)$ at $z = i, +2i$ (simple pole)

Res. of $f(z)$ at $z=a$ is a simple pole $\Big\} = \lim_{z\to a} (z-a) f(z)$

$a = i$

$$= \lim_{z\to +i} (z-i) \frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{z^2}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)}$$

$$= \frac{i^2}{2i(3i(-i))} = \frac{-1}{6i^2(-i)}$$

$$= \frac{-1}{6(-1)(-i)} = \frac{-1}{6i}$$

Res. of $f(z)$ at $z=2i$ is simple pole $\Big\} = \lim_{z\to 2i} \dfrac{(z-2i)\; z^2}{(z+i)(z-i)(z+2i)(z-2i)}$

$$= \frac{(2i)^2}{3i(i)(4i)} = \frac{-4}{12(-i)} = \frac{1}{3i}$$

② By Jordan Lemon

$$\int_{\Gamma} \frac{z^2 dz}{(z^2+1)(z^2+4)} = 0$$

$$\int_{C} \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_{-R}^{R} \frac{x^2 dx}{(x^2+1)(x^2+4)} \longrightarrow ③$$

③ $$\int_{C} \frac{z^2 dz}{(z^2+1)(z^2+4)} = 2\pi i \left[\text{Sum of the residue}\right]$$

$$= 2\pi i \left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

$$= 2\pi i \left[\frac{-3i+6i}{18i^2}\right]$$

$$= 2\pi i \left[\frac{3i}{18i^2}\right]$$

$$= \frac{6\pi \cancel{i^2}}{18 \cancel{i^2}}$$

$$= \frac{\pi}{3}$$

③ $$\int_{C} \frac{z^2 dz}{(z^2+1)(z^2+4)} = \frac{\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}$$

BHARATH INSTITUTE OF SCIENCE AND TECHNOLOGY

CONTINUOUSLEARNINGASSESSMEN TEST – I
U20MABT02- ADVANCED CALCULUS AND COMPLEX ANALYSIS
(SETB)

Date :22.11.2022
Academic Year/Semester 20222022/ODD
Duration :1:30Hours

Q.No.	PART–A(5X2=10)	Weightage	CO's	Bloom's Level
1.	Evaluate the double integral $\int_1^2 \int_x^{x^2} x\,y\,dy\,dx$.	2	CO1	2
2.	Evaluate $\int_1^a \int_1^b \frac{dx\,dy}{xy}$.	2	CO1	2
3.	Evaluate $\iint_s x^2y^2\,dx\,dy$ over the region bounded by the straight line $x=0,\ x=3, y=0,\ y=3$.	2	CO1	2
4.	Find the $grad\ \emptyset$ if $\emptyset = xyz$ at (1,1,1)	2	CO2	2
5.	Find the unit normal vector to the surface $x^2 - y^2 + z = 2$ at the point (1, -1, 2)	2	CO2	3

	PART–B(2X4=8)			
6.	(A). Find the area using double integral bounded by the lines $x=0,\ y=1\ and\ y=x$ **(OR)** (B). Find the area of first quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	4	CO1	3
7	(A).If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ **(OR)** (B) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, prove that $\nabla r^n = nr^{n-2}\vec{r}$	4	CO2	3
	PARTC(1X12=12)			
8.	(A) Change the order of integration to evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\ dy\,dx$. **(OR)** (B). Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using triple integral.	12	CO1	3

CO's	Weightage
CO1	22
CO2	08
Total	30

TERM - I

U20MABT02 - ADVANCED CALCULUS & COMPLEX ANALYSIS

CLA-1 - ANSWER KEY

Part - A

1. Evaluate the double Integral $\int_1^2 \int_x^{x^2} xy\, dx\, dy$: Ans: $27/8$

2. Evaluate $\iint_S x^2y^2\, dx\, dy$ over the region bounded by the line $x=0,3$, $y=0,3$

Ans: 81 Sq. units.

3. Evaluate $\int_1^a \int_1^b \frac{dx\,dy}{xy}$: Ans: $(\log a)(\log b)$

4. Find the grad ϕ if $\phi = xyz$ at $(1,1,1)$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (\vec{i})\,yz + \vec{j}(xz) + \vec{k}(xy)$$

at $(1,1,1)$,

$$= \vec{i} + \vec{j} + \vec{k}$$

5. Find the unit normal vector to the surface $x^2 - y^2 + z = 2$ at the pt $(1,-1,2)$.

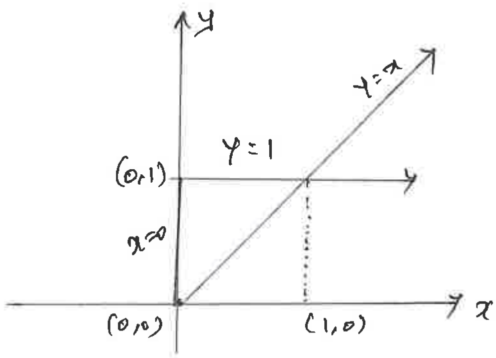
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} =$$

$$\nabla\phi = 2\vec{i} + 2\vec{j} + \vec{k}, \quad |\nabla\phi| = \sqrt{9} = 3$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

Part - B

6. (A) Find the area using double Integration bounded by straight line $x=0$, $y=1$ and $y=x$



$$= \int_{x=0}^{x=1} \int_{y=x}^{y=1} dy\, dx$$

$$= \int_0^1 [y]_x^1\, dx$$

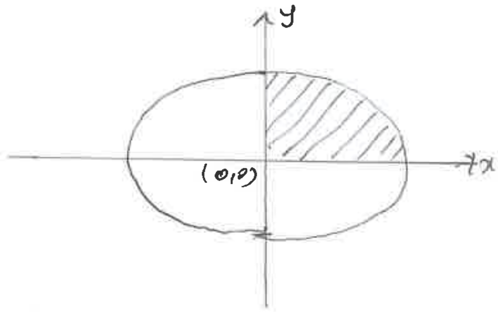
$$= \int_0^1 (1-x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1$$

$$= 1/2$$

6. (B). Find the area of first quadrant of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

let the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\boxed{x=\pm a}$, $\boxed{y=\pm b\sqrt{1-\frac{x^2}{a^2}}}$



$$\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$=\frac{\pi ab}{4}$$

7 (A) If $\vec{F}=x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$, find $\nabla.\vec{F}$ and $\nabla\times\vec{F}$.

$$\nabla.\vec{F}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(x^2\vec{i}+y^2\vec{j}+z^2\vec{k})$$

$$=2x+2y+2z$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 & y^2 & z^2\end{vmatrix}=0.$$

~~Part-B~~

8 (B) If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, Prove that $\nabla r^n=nr^{n-2}\vec{r}$.

Given $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

To Prove $\nabla r^n=$?

$$\nabla.r^n=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).r^n$$

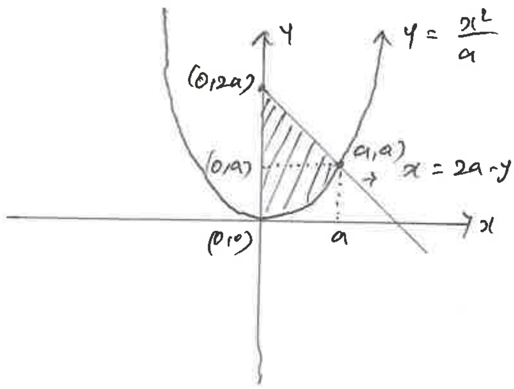
$$=n.r^{n-1}.\frac{\vec{r}}{r}=nr^{n-1}.r^{-1}.\vec{r}$$

$$=nr^{n-2}.\vec{r}$$

Part. C

8. (A) Chang of order of integration to evaluate.

$$\int_0^a\int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx.$$



$x=0$ in $y=2a$

$x=a$ in $y=a$

②

$$\int_0^a \int_0^{\sqrt{ya}} xy\, dx\, dy + \int_a^{2a} \int_0^{2a-y} xy\, dx\, dy$$

$$\Rightarrow \int_0^a \frac{y^3 a}{2}\, dy + \int_a^{2a} \frac{(2a-y)^2 xy}{2}\, dy$$

$$\Rightarrow \frac{a^4}{6} + \frac{a^4}{24}$$

$$\Rightarrow \frac{5a^4}{24}$$

(B) Find the volumn of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using triple integral.

ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\boxed{x = \pm a}$$

$$\boxed{y = \pm b\sqrt{1 - \frac{x^2}{a^2}}}$$

$$\boxed{z = \pm c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$$

$$8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dz\, dy\, dx$$

$$8c\int_0^a \left(\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\underbrace{1 - x^2/a^2}_{a^2} - \underbrace{y^2/b^2}_{x^2}}\; dy\right) dx.$$

$$\boxed{\sqrt{a^2 - x^2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}(x/a)}$$

$\frac{\pi abc}{3}$ sq. units

$$\iint_{S_4} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} (zx)\,dz\,dx$$

$$= \int_0^a\int_0^c (zx)\,dz\,dx$$

$$= \int_0^a \left[\frac{z^2}{2}x\right]_0^c dx$$

$$= \int_0^a \frac{c^2}{2}.x\,dx$$

$$= \frac{a^2c^2}{4} \;//$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} (c^2 - xy)\,dy\,dx$$

$$= \int_0^a \left(c^2y - \frac{xy^2}{2}\right)_0^b dx$$

$$= \int_0^a \left(c^2b - \frac{xa^2}{2}\right) dx$$

$$= \left(c^2bx - \frac{x^2b^2}{2\,2}\right)_0^a$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_{S_6} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=a}\int_{y=0}^{y=b} (xy)\,dy\,dx$$

$$= \int_0^a \left(x\frac{y^2}{2}\right)_0^b dx$$

$$= \int_0^a \frac{b^2x}{2}\,dx$$

$$= \frac{b^2}{2}\left(\frac{x^2}{2}\right)_0^a$$

$$= \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,ds = a^2cb - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + b^2ca - \cancel{\frac{c^2a^2}{4}} + \cancel{\frac{a^2c^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$= a^2cb + b^2ca + abc^2$$

$$\boxed{\therefore \iint_S \vec{F}\cdot\vec{n}\,ds = abc(a+b+c)}$$

$\therefore$ Gauss Divergence Theorem is Verified.

Stokes Theorem:

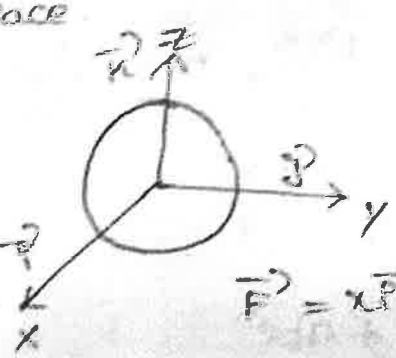
Let 'S' be an open Surface bounded by a closed curve 'c'. If $\vec{F}$ is a Vector Point function acting inside and on the Surface 'S' then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, ds$$

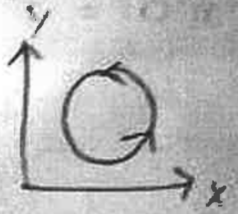
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, ds$$

'c' is a Simple closed Curve in Anticlockwise direction.

'n' is unit outward normal Vector to the given Surface



$\vec{F} = x\vec{i} + y\vec{j}$



$\vec{F} = x\vec{i} + y\vec{j} + 0\vec{k}$

$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_C P\,dx + Q\,dy \quad (1)$$

Green's Theorem is a Particular case of Stokes theorem when the 3D open object can be converted into 2D image. $\therefore \vec{k} = 0$

Let, $\vec{F} = P\vec{i} + Q\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$\vec{F}\cdot d\vec{r} = (P\vec{i} + Q\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$

$$\boxed{\vec{F}\cdot d\vec{r} = P\,dx + Q\,dy}$$

To find $\nabla\times\vec{F}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ P & Q & 0 \end{vmatrix}$$

$$\nabla\times\vec{F} = \vec{i}\left(0-\frac{dQ}{dz}\right)-\vec{j}\left(0-\frac{dP}{dz}\right)+\vec{k}\left(\frac{dQ}{dx}-\frac{dP}{dy}\right)$$

$\hat{n} = \vec{k}$ ($\hat{n}$ is the unit outward normal vector to xy plane which is parallel to z-axis)

$$[\nabla\times\vec{F}]\cdot\hat{n} = \frac{dQ}{dx}-\frac{dP}{dy} \quad (\hat{k}\cdot\hat{k}=1)$$

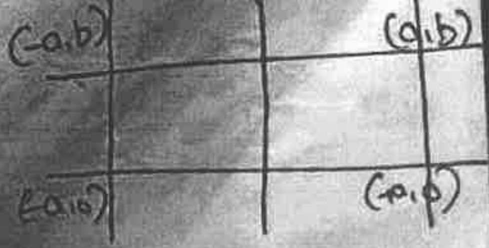
$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$$ → Stokes theorem (3D open object)

$$\Rightarrow \oint_C P\,dx+Q\,dy = \iint_{R \text{ on } xy \text{ plane}} \left(\frac{dQ}{dx}-\frac{dP}{dy}\right)dx\,dy$$ → Green's theorm 2D image on xy-plane

① Verify Stokes Theorem $\vec{F}=(x^2+y^2)\vec{i}-2xy\vec{j}$ taken around the rectangle $x=\pm a$, $y=0, y=b$

Sol:- $$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix}$$



$$\nabla\times\vec{F} = \vec{i}\left[(0) - \frac{d}{dz}(-2xy)\right] - \vec{j}\left[0 - \frac{d}{dz}(x^2+y^2)\right] + \vec{k}\left[\frac{d}{dx}(-2xy) - \frac{d}{dy}(x^2+y^2)\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y - 2y)$$

$$= (-4y)\vec{k}$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds = \int_{x=-a}^{x=a}\left[\int_{y=0}^{y=b}(-4y)\,dy\right]dx$$

$$= \int_{-a}^{a}\left[-\frac{\cancel{4}^{2}y^2}{\cancel{2}}\right]_0^b dx$$

$$= \int_{-a}^{a} -2b^2\,dx$$

$$= -2b^2[x]_{-a}^{a}$$

$$= -2b^2[a-(-a)]$$

$$= 2a(-2b^2) = -4ab^2 \,//$$

$$\oint_{C:} \vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx + (-2xy)\,dy$$

$$\oint_{C:-a}^{a} \vec{F}\cdot d\vec{r} = \oint_{-a}^{a} x^2+y^2$$

$x = -a$

$\boxed{dx = 0}$

$$= \int_{x=-a}^{x=a} (x^2+0)\,dx + -2x(0)\,dy$$

$$= \int_{-a}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{3}(a^3+a^3) = \frac{2a^3}{3}$$

$$\oint_{y=0}^{y=b} \vec{F}\cdot d\vec{r} = \int_0^b (x^2+y^2)dx - 2xy\,dy$$

$y=0$, $x=a \Rightarrow \boxed{dx=0}$

$$= \int_0^b 0 - 2ay \quad = -2a\frac{b^2}{2} = -ab^2$$

$$= \int_{x=a}^{x=-a} (x^2+y^2)dx - 2xy\,dy$$

$x=a$, $y=b$ $\boxed{dy=0}$

$$= \int_a^{-a} (x^2+b^2)dx - 0 \quad = \left[\frac{x^3}{3}\right]_a^{-a} + b^2[x]_a^{-a}$$

$$= -\frac{a^3}{3} - \frac{a^3}{3} - b^2(-2a)$$

$$= -\frac{2a^3}{3} + 2ab^2.$$

$$= \int_b^0 (x^2+y^2)dx - 2xy\,dy$$

$y=b$, $x=-a$ $\boxed{dx=0}$

$$= \int_b^0 -2(-a)y\,dy \quad = 2a\left[\frac{y^2}{2}\right]_b^0$$

$$= 2a\left(0-\frac{b^2}{2}\right) \quad = -\frac{\cancel{2}ab^2}{\cancel{2}} = -ab^2$$

$$C_1+C_2+C_3+C_4 = -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 + \cancel{\frac{2a^3}{3}} - \frac{2ab^2}{2}$$

$$= -4ab^2$$

Laplace Transform Technique

Laplace

Let $F(t)$ be a function defined on the interval $[0,\infty)$. $[0 \le t < \infty]$ then

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt \;;\; t \ge 0$$

Where,

L - Laplacian operator

S - Parameter

Application of L.T.T :-

① To Solve System of Ordinary differential equations.

② Evaluate System reliability.

③ Used in Digital Signal processing.

④ Wireless Network.

① Find $L[1] = \int_0^\infty e^{-st}(1)\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1)$$

$$= \frac{1}{s}$$

* $L[e^{at}]$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(t) = e^{at}$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\,dt = \int e^{-(s-a)t}\,dt$$

$$\int e^{-x}dx = \frac{e^{-x}}{-1}$$

$$\int e^{-2x}dx = \frac{e^{-2x}}{-2}$$

$$= \left(\frac{e^{-(s-a)t}}{-(s-a)}\right)_0^\infty = \frac{-1}{(s-a)}\left(e^{-(s-a)t}\right)_0^\infty$$

$$= \frac{-1}{s-a}\left[0 - e^{-(s-a)0}\right] = \frac{-1}{s-a}(0-e^{0})$$

$$= \frac{-1}{s-a}(-1) = \frac{1}{s-a}$$

$$L[e^{at}] = \frac{1}{s-a} \qquad s-a>0 \qquad s>a$$

$$L[e^{-at}] = \frac{1}{s-(-a)} = \frac{1}{s+a}$$

(*) Find $L[t^n]$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$f(t) = t^n$$

$$L[t^n] = \int_{t=0}^{t=\infty} e^{-st}\, t^n\, dt$$

Let $st = u \Rightarrow t = \frac{u}{s}$

$$s\,dt = du$$

$$dt = \frac{du}{s}$$

if $t=0 \Rightarrow u=0$

$t=\infty \Rightarrow u=\infty$

$$L[t^n] = \int_{u=0}^{u=\infty} e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^{\infty} \frac{e^{-u} u^n\, du}{s^n . s}$$

$$= \int_0^{\infty} \frac{e^{-u} . u^n\, du}{s^{n+1}}$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-u} . u^n . du$$

$n = n-1$
$n+1 = n$

**

$$\boxed{\text{Gamma } \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx}$$

Comparing

$$= \frac{1}{s^{n+1}} \cdot \int_0^{\infty} e^{-x} x^{(n+1)-1} dx = \frac{\Gamma(n+1)}{s^{n+1}}$$

$\circledast \; L[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}}$ If n - rational (fractional value)

$= \dfrac{n!}{s^{n+1}} \rightarrow$ integer

$\circledast$

$\rightarrow \Gamma(n+1) = n\Gamma n$ $\quad \Gamma\frac{1}{2} = \sqrt{\pi}$

$\rightarrow \Gamma(3/2) = \Gamma(1/2+1) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

$\rightarrow \Gamma 5/2 = \Gamma\frac{3}{2}+1 = \frac{3}{2}\Gamma\frac{3}{2} = \frac{3}{2}\Gamma\frac{1}{2}+1 = \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

$= \frac{3}{4}\sqrt{\pi}$

$\rightarrow \cancel{\Gamma 7/2 = \frac{7}{2}\Gamma\frac{7}{2} = \frac{7}{2}\Gamma\frac{5}{2}+1 = \frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2}}$

$\cancel{= \frac{35}{4}\Gamma\frac{3}{2}+1 = \frac{35}{4}\cdot\frac{3}{2}\Gamma\frac{3}{2}}$

If n - Integer

$\Gamma n+1 = n\Gamma n = n!$

$\Gamma 2 = \Gamma 1+1 = 1\Gamma 1 = 1$

$\Gamma 3 = \Gamma 2+1 = 2\Gamma 2 = 2\Gamma 1+1 = 2\cdot 1\Gamma 1 = 2$

$\Gamma 4 = \Gamma 3+1 = 3\Gamma 3 = 3\Gamma 2+1 = 3\,2\,\Gamma 2 = 6 = 3!$

$\Gamma 5 = \Gamma 4+1 = 4!$

(*) $L[t^0] = L[1] = \frac{1}{s}$

$L[t^1] = L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

$L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$L[t^4] = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$

$L[t^5] = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$

Problems

① Find $L[t^{1/2}]$ $n = 1/2$

$L[t^{1/2}] = \frac{\Gamma 1/2+1}{s^{1/2+1}} = \frac{\cancel{\sqrt{\pi}}\ \frac{1}{2}\Gamma 1/2}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$

② Find $L[t^{-1/2}]$ $n = -1/2$

$L[t^{-1/2}] = \frac{\Gamma -1/2+1}{s^{-1/2+1}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$

③ Find $L[e^{2t} + 2e^{-3t} - 4\sinh 2t + 5\cosh t + 7\sin 3t]$

Sol:-

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\boxed{\begin{aligned} L[e^{at}] &= \frac{1}{s-a} \\ L[e^{-at}] &= \frac{1}{s+a} \\ L[\sin at] &= \frac{a}{s^2+a^2} \end{aligned}}$$

$$\boxed{\begin{aligned} L[\cos at] &= \frac{s}{s^2+a^2} \\ L[\sinh at] &= \frac{a}{s^2-a^2} \\ L[\cosh at] &= \frac{s}{s^2-a^2} \end{aligned}}$$

$$= L[e^{2t}] + L[2e^{-3t}] + L[-4\sinh 2t] + L[5\cosh t] + L[7\sin 3t]$$

$$= \frac{1}{s-2} + 2\left[\frac{1}{s+3}\right] - 4\left(\frac{2}{s^2-4}\right) + 5\left(\frac{s}{s^2-1}\right) + \frac{21}{s^2+9}$$

$$= \frac{1}{s-2} + \frac{2}{s+3} - \frac{8}{s^2-4} + \frac{5s}{s^2-1} + \frac{21}{s^2+9}$$ //

(5) Find $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$

$$= L[a\sqrt{t}] + L\left[\frac{b}{\sqrt{t}}\right] + L[c]$$

$$= aL[t^{1/2}] + bL\left[\frac{1}{t^{1/2}}\right] + L[c]$$

$$= a\frac{\sqrt{\pi}}{2s^{3/2}} + b\sqrt{\frac{\pi}{s}} + cL[1]$$

$$= \frac{a\sqrt{\pi}}{2s^{3/2}} + b\sqrt{\frac{\pi}{s}} + \frac{c}{s} \quad //$$

(6) $L[t^4+3t^3-5t^2+2t-4]$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

$$= L[t^4] + L[3t^3] - L[5t^2] + L[2t] - L[4]$$

$$= \frac{4!}{s^{4+1}} + 3\left(\frac{3!}{s^{3+1}}\right) - 5\left(\frac{2!}{s^{2+1}}\right) + 2\left(\frac{1!}{s^{1+1}}\right) + \left(\frac{-4(1!)}{s^{1+1}}\right)$$

$$= \frac{24}{s^5} + \frac{18}{s^4} - \frac{10}{s^3} + \frac{2}{s^2} - \frac{4}{s}$$

(7) $L[t^3+2t^2-t-10]$

$$= \frac{3!}{s^4} + 2\left(\frac{2!}{s^3}\right) - \frac{11}{s^2} - \frac{10}{s}$$

$$= \frac{6}{s^4} + \frac{4}{s^3} - \frac{11}{s^2} - \frac{10}{s}$$

⑧ $L[(t-1)^3]$

$= L[t^3 - 3t^2 + 3t - 1]$

$= L[t^3] - 3L[t^2] + 3L[t] - L[1]$

$= \dfrac{3!}{s^{3+1}} - 3\left(\dfrac{2!}{s^{2+1}}\right) + 3\left(\dfrac{1!}{s^{1+1}}\right) - \dfrac{1}{s}$

$= \dfrac{6}{s^4} - \dfrac{6}{s^3} + \dfrac{3}{s^2} - \dfrac{1}{s}$

⑨ $L[(t+1)^3] = L[t^3 + 3t^2 + 3t + 1]$

$= \dfrac{3!}{s^{3+1}} + 3\left(\dfrac{2!}{s^{2+1}}\right) + 3\left(\dfrac{1!}{s^{1+1}}\right) - \dfrac{1}{s}$

$= \dfrac{6}{s^4} + \dfrac{6}{s^3} + \dfrac{3}{s^2} + \dfrac{1}{s}$

Formula:

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

⑩ $L[\sin^2 3t] = L\left[\dfrac{1-\cos 6t}{2}\right]$

$= \dfrac{1}{2} L[1 - \cos 6t]$

$= \dfrac{1}{2}\left[L[1] - L[\cos 6t]\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{s^2+6^2}\right] = \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{s^2+36}\right]$

(11) $L[\cos^2 4t] = L\left[\frac{1+\cos 8t}{2}\right]$

$= \frac{1}{2}\left[L[1+\cos 8t]\right]$

$= \frac{1}{2}\left[L[1] + L[\cos 8t]\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$

Formula:

$\sin(A+B) = \sin A \cos B + \cos A \sin B \rightarrow (1)$

$\sin(A-B) = \sin A \cos B - \cos A \sin B \rightarrow (2)$

$\cos(A+B) = \cos A \cos B - \sin A \sin B \rightarrow (3)$

$\cos(A-B) = \cos A \cos B + \sin A \sin B \rightarrow (4)$

Add (1) + (2)

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$

(12) $L[\sin 5t \cos 3t]$

$= L\left[\frac{\sin(5t+3t) + \sin(5t-3t)}{2}\right]$

$= \frac{1}{2} L[\sin 8t + \sin 2t]$

$= \frac{1}{2}\left[L[\sin 8t] + L[\sin 2t]\right]$

$= \frac{1}{2}\left[\frac{8}{s^2+64} + \frac{2}{s^2+4}\right]$

(13) $L[\sin 4t \cdot \sin 6t]$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$L[\sin 4t \cdot \sin 6t] = \frac{1}{2}[\cos[4t-6t] - \cos[4t+6t]]$

$= \frac{1}{2}[+\cos 2t - \cos 10t]$

$= \frac{1}{2}\left[\frac{+s}{s^2+4} - \frac{s}{s^2+100}\right]$

Shifting Property:

Stationar:

If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

$L[e^{-at} f(t)] = F(s+a)$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st}[e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(st-at)} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

$$L\left[e^{-at}f(t)\right] = \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

(14) $L\left[e^{2t}\cosh 3t\right] = L\left[e^{at}f(t)\right]$

$a = 2$, $f(t) = \cosh 3t$

$$L[f(t)] = L[\cosh 3t]$$

$$= \left[\frac{s}{s^2-9}\right] = F(s)$$

$$L\left[e^{2t}\cosh 3t\right] = \left[\frac{s}{s^2-9}\right]_{s\to s-2} = \frac{s-2}{(s-2)^2-9}$$

(15) $L\left[e^{-3t}\sin 4t\right] = L\left[e^{-at}f(t)\right]$

$a=3$, $f(t) = \sin 4t$

$$L[f(t)] = L[\sin 4t]$$

$$F(s) = \frac{4}{s^2+16}$$

$$L\left[e^{-3t}\sin 4t\right] = \cancel{L}\left[F(s)\right]_{s\to s+a}$$

$$= \frac{4}{s^2+16}$$

$$= \frac{4}{(s+3)^2+16}$$

Scaling Property of L.T.T:

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof:

$$L[f(t)] = F(s)$$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

Let $at = u$

if $t = 0 \Rightarrow u = 0$

if $t = \infty \Rightarrow u = \infty$

$at = u$

$a\,dt = du$

$dt = \frac{du}{a}$

$$L[f(u)] = \int_{u=0}^{u=\infty} e^{-s\left(\frac{u}{a}\right)} f(u) \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s\left(\frac{u}{a}\right)} f(u)\,du$$

$$L[f(u)] = L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Differentiation of LTT:

If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$

Proof

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$n = 1$

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\,dt\right]$$

$$= \int_0^\infty \left[\frac{d}{ds}(e^{-st})\right] f(t)\,dt$$

$$\frac{d}{ds}(e^{-st})$$

$$= e^{-st}\frac{d}{ds}(-st)$$

$$= e^{-st}(-t)$$

$$= \int_0^\infty e^{-st}(-t)\, f(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t) f(t)\,dt$$

$$\Rightarrow \frac{d}{ds}[F(s)] = -\int_0^\infty e^{-st}[t f(t)]\,dt$$

$$= -L[t f(t)]$$

$$= L[t f(t)] = \frac{-d}{ds}[F(s)]$$

$n=2$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

$n=3$

$$L[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[F(s)]$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

(16) $L[t^2 \cos 2t]$

Compare $L[t^n f(t)]$

$n=2 \quad f(t) = \cos 2t$

$$F(s) = L[f(t)] = L[\cos 2t]$$

$$F(s) = \frac{s}{s^2+4}$$

$$L\left[t^n f(t)\right] = (-1)^2 \frac{d^2}{ds^2}\left[F(s)\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{s}{s^2+4}\right]\right]$$

$$\frac{d}{ds}\left[\frac{s}{s^2+4}\right] = \frac{(s^2+4)(1) - s(2s)}{(s^2+4)2}$$

$$= \frac{s^2+4-2s^2}{(s^2+4)2} = \frac{4-s^2}{(s^2+4)^2}$$

$$\frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right] = \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-2s) - \cancel{4s}(4-s^2)\,2(s^2+4)(2s)}{\left[(s^2+4)^2\right]^2}$$

$$= \frac{-2s(s^2+4)^2 - 4s(4-s^2)(s^2+4)}{(s^2+4)^4}$$

$$= -2s(s^2+4)\left[\frac{(s^2+4)+2(4-s^2)}{(s^2+4)^4}\right]$$

$$= -2s\left[\frac{(s^2+4)+2(4-s^2)}{(s^2+4)^3}\right] //$$

(F) $L[t^2 \cosh 3t]$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

$n=2$

$$L[f(t)] = L[\cosh 3t]$$

$$F(s) = \frac{s}{s^2-3^2} = \frac{s}{s^2-9}$$

$n=2$

$$(-1)^2 \frac{d^2}{ds^2}[F(s)] = \frac{d^2}{ds^2}\left[\frac{s}{s^2-9}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2-9}\right)\right] \rightarrow ②$$

$$\frac{d}{ds}\left(\frac{s}{s^2-9}\right) = \frac{(s^2-9)(1) - s(2s)}{(s^2-9)^2}$$

$$= \frac{s^2-9-2s^2}{(s^2-9)^2}$$

$$= \frac{-9-s^2}{(s^2-9)^2}$$

$$= -\left[\frac{s^2+9}{(s^2-9)^2}\right] \rightarrow ③$$

$$\frac{d}{ds}\left[-1\left(\frac{s^2+9}{(s^2-9)^2}\right)\right] = -\frac{d}{ds}\left[\frac{s^2+9}{(s^2-9)^2}\right]$$

$$= \frac{(s^2-9)^2(2s) - (s^2+9)\,2(s^2-9)2s}{((s^2-9)^2)^2}$$

$$= -\left[\frac{(s^2-9)^2(2s) - 4s(s^2+9)(s^2-9)}{(s^2-9)^4}\right]$$

$$= -(s^2-9)\left[\frac{(s^2-9)(2s) - 4s(s^2+9)}{(s^2-9)^4}\right]$$

$$= -\left[\frac{2s(s^2-9) - 4s(s^2+9)}{(s^2-9)^3}\right]$$

$$= -\left[\frac{2s^3 - 18s - 4s^3 - 36s}{(s^2-9)^3}\right]$$

$$= -\left[\frac{-2s^3 + 0 - 54s}{(s^2-9)^3}\right]$$

$$= \frac{54s + 2s^3}{(s^2-9)^3} \quad //$$

Inverse Laplane Trans-form:

If $L[f(t)] = F(s)$ then to find $f(t)$ → ①

Take L^{-1} on both sides

$$L^{-1}\{L[f(t)]\} = L^{-1}[F(s)]$$

$$L^{-1}L[f(t)] = L^{-1}[F(s)]$$

$$\boxed{f(t) = L^{-1}[F(s)]}$$

To find $f(t)$ then $L^{-1}[F(s)]$ given:

Note:

① $f(t) \xrightarrow[\text{Transform}]{\text{Laplace}} L[f(t)] = F(s)$

② $F(s) \xrightarrow[\text{Laplace}]{\text{Inverse}} L^{-1}[F(s)] \xrightarrow[f(t)]{\text{To find}} f(t)$

Function $f(t)$	Laplace Transform	$L^{-1}[F(s)]$	$f(t)$
$f(t)=c$	$L[c] = c/s$	$L^{-1}\left[\frac{c}{s}\right] = c$	c
$f(t)=1$	$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$	1
e^{at}	$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$	e^{at}
e^{-at}	$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$	e^{-at}
$\sin at$	$L[\sin at] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$	$\sin at$
$\cos at$	$L[\cos at] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$	$\cos at$
$\sinh at$	$L[\sinh at] = \frac{a}{s^2-a^2}$	$L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$	$\sinh at$
$\cosh at$	$L[\cosh at] = \frac{s}{s^2-a^2}$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$	$\cosh at$
t^n	$L[t^n] = \frac{n!}{s^{n+1}}$	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$	t^n
t^2	$L[t^2] = \frac{2!}{s^{2+1}}$	$L^{-1}\left[\frac{2!}{s^3}\right] = t^2$	t^2

$t^{1/2}$	$L[t^{1/2}] = \frac{\sqrt{\pi}}{2s^{3/2}}$	$L^{-1}\left[\frac{\sqrt{\pi}}{2s^{3/2}}\right] = t^{1/2}$	$t^{1/2}$
$t^{-1/2}$	$L[t^{-1/2}] = \sqrt{\frac{\pi}{s}}$	$L^{-1}\left[\sqrt{\frac{\pi}{s}}\right] = t^{-1/2}$	$t^{-1/2}$

Problems

① $L^{-1}\left[\frac{3s+2}{s^2-4}\right] = L^{-1}[F(s)]$

Sol:-

$$= L^{-1}\left[\frac{3s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right]$$

$a=2$

$$\Rightarrow 3\cosh 2t + \sinh 2t = f(t)$$

Partial Fractions: (Nr degree < Dr. Degree)

Type-1

$$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

Type-2

$$\frac{1}{(s+a)(s+b)^2} = \frac{A}{(s+a)} + \frac{B}{(s+b)} + \frac{C}{(s+b)^2}$$

Type-3

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

② $L^{-1}\left[\dfrac{s+2}{s(s+4)(s-9)}\right]$

Sol:- $\dfrac{s+2}{s(s+4)(s-9)} = \dfrac{A}{s} + \dfrac{B}{(s+4)} + \dfrac{C}{s-9}$

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A(s+4)(s-9) + B(s)(s-9) + C(s)(s+4)}{s(s+4)(s-9)}$$

Put $s=-4$

$$-4+2 = A(-4+4)(-4-9) + B(-4)(-4-9) + C(-4)(-4+4)$$

$$-2 = 0 + 52B$$

$\cancel{B = \frac{52}{2}}$ $\quad B = \dfrac{-2}{52} \Rightarrow \boxed{B = \dfrac{-1}{26}}$

$s=9$ $\quad 11 = C(9(9+4))$

$$11 = 9(13)C$$

$$C = \frac{11}{117}$$

$s=0$ $\quad 2 = -36A$

$$A = \frac{-2}{36} = \frac{-1}{18}$$

$$\frac{s+2}{s(s+4)(s-9)} = \frac{-1/18}{s} + \frac{-1/26}{s+4} + \frac{11/117}{s-9}$$

$$L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right] = \frac{-1}{18} L^{-1}\left[\frac{1}{s}\right] - \frac{1}{26} L^{-1}\left[\frac{1}{s+4}\right] + \frac{11}{117} L^{-1}\left[\frac{1}{s-9}\right]$$

$$= \frac{-1}{18}(1) - \frac{1}{26}e^{-4t} + \frac{11}{117}e^{9t}$$ //

③ Find $L^{-1}\left[\frac{s}{(s-4)(s+5)}\right]$

Sol:

$$\frac{s}{(s-4)(s+5)} = \frac{A}{(s-4)} + \frac{B}{(s+5)}$$

$$s = A(s+5) + B(s-4)$$

Put $s=-5$

$$-5 = 0 + B(-9)$$

$$\boxed{B = \frac{5}{9}}$$

Put $s=4$

$$4 = 9A + 0$$

$$9A = 4 \Rightarrow \boxed{A = \frac{4}{9}}$$

$$\frac{s}{(s-4)(s+5)} = \frac{4/9}{(s-4)} + \frac{5/9}{(s+5)}$$

$$L^{-1}\left[\frac{s}{(s-4)(s+5)}\right] = L^{-1}\left[\frac{4}{9(s-4)}\right] + L^{-1}\left[\frac{5}{9(s+5)}\right]$$

$$= \frac{4}{9}L^{-1}\left[\frac{1}{s-4}\right] + \frac{5}{9}\left[\frac{1}{s+5}\right]$$

$$= \frac{4}{9}e^{4t} + \frac{5}{9}e^{-5t}$$ //

④ $L^{-1}\left[\frac{S+1}{S(S-2)(S-3)}\right]$

$$\frac{S+1}{S(S-2)(S-3)} = \frac{A}{S} + \frac{B}{(S-2)} + \frac{C}{S-3}$$

$$S+1 = A(S-2)(S-3) + B(S)(S-3) + C(S)(S-2)$$

put $S=2$

$$3 = B(2)(2-3)$$

$$3 = B(2)(-1)$$

$$-2B = 3$$

$$\boxed{B = -\frac{3}{2}}$$

put $S=3$

$$4 = C(3)(3-2)$$

$$4 = 3C$$

$$\boxed{C = \frac{4}{3}}$$

put $S=0$

$$1 = A(-2)(-3)$$

$$6A = 1$$

$$\boxed{A = \frac{1}{6}}$$

$$\frac{S+1}{S(S-2)(S-3)} = \frac{1}{6S} + \frac{(-3)}{2(S-2)} + \frac{4}{3(S-3)}$$

$$L^{-1}\left(\frac{S+1}{S(S-2)(S-3)}\right) = \cancel{\tfrac{1}{6}}\, L^{-1}\left[\frac{1}{6S} - \frac{3}{2(S-2)} + \frac{4}{3(S-3)}\right]$$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{S}\right] - \frac{3}{2}L^{-1}\left[\frac{1}{S-2}\right] + \frac{4}{3}\left[\frac{1}{S-3}\right]$$

$$= \frac{1}{6} - \frac{3}{2}e^{2t} + \frac{4}{3}e^{3t}$$

(5) $L^{-1}\left[\frac{4s^2+5s-3}{(s-1)^2(s+2)}\right]$

Sol:-

$$\frac{4s^2+5s-3}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$4s^2+5s-3 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$s=1$

$$4+5-3 = A(0) + 3B$$

$$3B = 6$$

$$\boxed{B=2}$$

$s=-2$

$$4(4)+5(-2)-3 = A(0) + B(0) + C(-2-1)^2$$

$$16-10-3 = 9C$$

$$3 = 9C \Rightarrow \boxed{C = \frac{1}{3}}$$

$s=0$

$$-3 = A(-1)(2) + B(2) + C$$

$$-3 = -2A+2B+C$$

$$-3 = -2A + 4 + \frac{1}{3}$$

$$\cancel{-2}A = \frac{\cancel{-22}\ 11}{3}$$

$$\boxed{A = \frac{11}{3}}$$

$$\boxed{\begin{array}{l} L[t] = \frac{1}{s^2} \\ L^{-1}\left[\frac{1}{s^2}\right] = te^t \end{array}}$$

$$L^{-1}\left[\frac{4s^2+5s-3}{(s-1)^2(s+2)}\right] = L^{-1}\left[\frac{11}{3(s-1)} + \frac{2}{(s-1)^2} + \frac{1}{3(s+2)}\right]$$

$$= \frac{11}{3}e^t + 2te^t + \frac{1}{3}e^{-2t}$$

Convolution of Laplace transform:

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

The $*$ denotes convolution operator.

Note:

$$L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$$

$$L^{-1}[F(s) \pm G(s)] = L^{-1}[F(s)] \pm L^{-1}[G(s)] \rightarrow ①$$

If $L[f(t)] = F(s) \Rightarrow f(t) = L^{-1}[F(s)]$

$L[g(t)] = G(s) \Rightarrow g(t) = L^{-1}[G(s)]$

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$= f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

Problems

① Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using Convolusion

Sol:- $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2} \cdot \dfrac{s}{s^2+b^2}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt \hookrightarrow [f(t) * g(t)]$$

$f(t) = \cos at$	$f(u) = \cos au$
$g(t) = \cos bt$	$g(u) = \cos b(t-u)$

$$= \int_0^t f(u)\, g(t-u)\,du$$

$$= \int_0^t \cos au \, . \cos b(t-u)\,du$$

$$= \int_0^t \cos au \, . \cos (bt-bu)\,du$$

$$\cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-bt+bu) \right]$$

$$= \frac{1}{2}\int_0^t \left[\cos(a-b)u+bt + \cos(a+b)u-bt \right]$$

$$= \frac{1}{2}\left[\frac{\sin(a+b)u-bt}{a+b} + \frac{\sin(a-b)u+bt}{a-b} \right]_0^t$$

$$= \frac{1}{2}\left[\left[\frac{\sin(a+b)t-bt}{a+b} - \frac{\sin(-bt)}{a+b} \right] + \left[\frac{\sin(a-b)t+bt}{a-b} - \frac{\sin bt}{a-b} \right] \right.$$

$$= \frac{1}{2}\left[\frac{\sin at}{a+b} + \frac{\sin bt}{a+b} + \frac{\sin at}{a-b} - \frac{\sin bt}{a-b} \right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a+b} + \frac{1}{a-b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right) \right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{a-\cancel{b}+a+\cancel{b}}{a^2-b^2}\right) + \sin bt\left(\frac{\cancel{a}-b-\cancel{a}-b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin at\,(2a)}{a^2-b^2} + \frac{\sin bt\,(-2b)}{a^2-b^2}\right]$$

$$= \frac{a\sin at}{a^2-b^2} - \frac{b\sin bt}{a^2-b^2}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2} \quad //$$

② Find $L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right]$

Sol:- $L^{-1}\left[\frac{s}{s^2+25}\cdot\frac{s}{s^2+25}\right]$

$$= L^{-1}\left[\frac{s}{s^2+25}\right] * L^{-1}\left[\frac{s}{s^2+25}\right]$$

$$\left\{\begin{aligned} &= f(t) * g(t) \\ &= \int_0^t f(u)\,g(t-u)\,du \end{aligned}\right\} = \cos 5t * \cos 5t$$

$f(t) = \cos 5t$ | $g(t) = \cos 5t$

$f(u) = \cos 5u$ | $g(t-u) = \cos 5(t-u)$

$$= \int_0^t \cos 5u \,.\, \cos(5t-5u)\,du$$

$$\left\{\cos A\,.\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]\right\}$$

$$= \frac{1}{2}\left[\int_0^t \cos(\cancel{5u}+5t-\cancel{5u}) + \cos(5u-5t+5u)\right]du$$

$$= \frac{1}{2}\left[\int_0^t \cos 5t + \cos(10u-5t)\right]du$$

$$= \frac{1}{2}\left[\int_0^t \cos 5t\, du + \int_0^t \cos(10u-5t)\, du\right]$$

$$= \frac{1}{2}\left[\cos 5t\,(u)_0^t + \left(\frac{\sin(10u-5t)}{10}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t\cos 5t + \frac{\sin(10t-5t)}{10} - \frac{\sin(-5t)}{10}\right]$$

$$= \frac{1}{2}\left[t\cos 5t + \frac{\sin 5t}{10} + \frac{\sin 5t}{10}\right]$$

$$= \frac{1}{2}\left[t\cos 5t + \frac{1}{5}\sin 5t\right] //$$

③ $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$

Sol:

$$= L^{-1}\left[\frac{s}{(s^2+4)} \cdot \frac{s}{(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+4}\right] * \left[\frac{s}{s^2+4}\right]$$

$$= f(t) * g(t) \qquad = \cos 2t * \cos 2t$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$f(t) = \cos 2t$ | $g(t) = \cos 2t$

$f(u) = \cos 2u$ | $g(t-u) = \cos 2(t-u)$

$$= \int_0^t \cos 2u \cdot \cos 2(t-u)\, du$$

$$= \int_0^t \cos 2u \cdot \cos(2t-2u)\, du$$

$$\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$= \frac{1}{2}\left[\int_0^t \cos(\cancel{2u}+2t-\cancel{2u}) + \cos(2u-2t+2u)\right] du$$

$$= \frac{1}{2}\left[\int_0^t \cos 2t\, du + \int_0^t \cos(4u-2t)\, du\right]$$

$$= \frac{1}{2}\left[\cos 2t\,(u)_0^t + \left(\frac{\sin(4u-2t)}{4}\right)_0^t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{4} - \frac{\sin(-2t)}{4}\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right] \quad //$$

(4) $L^{-1}\left[\frac{1}{s(s+5)}\right]$

$$= L^{-1}\left[\frac{1}{s} \cdot \frac{1}{s+5}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s+5}\right]$$

$$= f(t) * g(t)$$

$$= 1 * e^{-5t} = e^{-5t} * 1$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t 1 \cdot e^{-5(t-u)}\, du$$

$$L^{-1}\left[e^{-at}\right] = \frac{1}{s+a}$$

$$f(t) = e^{-5t}$$

$$= \int_0^t e^{-5u} \cdot 1\, du$$

$$= \left(\frac{e^{-5u}}{-5}\right)_0^t =$$

$$= -\frac{1}{5}\left(e^{-5t} - e^{-0}\right)$$

$$= -\frac{1}{5}\left(e^{-5t} - 1\right)$$

$$= \frac{1}{5} - \frac{e^{-5t}}{5}$$

$$= \frac{1 - e^{-5t}}{5}$$

//

4. Analytic Functions

(1) $z = x + iy$

(2) $F(z) = u + iv$

(3) $\omega = F(z)$

(4) $\omega = u + iv$

(5) $f(z) = u + iv$

(6) $f'(z) = u_x + iv_x$ Diff. w.r.to 'x'

(7) $f'(z) = u_y + iv_y$ Diff. w.r.to 'y'

$$f'(z) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

(8) $f(z) = u + iv$

$$= u(x,y) + iv(x,y)$$

Cauchy - Riemann Equation:

$$f(z) = u + iv$$

$$f'(z) = u_x + iv_x$$

$$f'(z) = u_y + iv_y$$

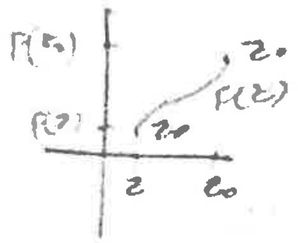
C-R Equation

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}} \begin{aligned} &\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ &\rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{aligned} \Bigg\} \text{ } f(z) \text{ is Analytic Function.}$$

Derivative of $F(z)$

A function $F(z)$ is differentiable at $z = z_0$ then the derivative of $F(z)$ is given by

$$F'(z) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} \text{ exists}$$



It is denoted by $\frac{d}{dz} F(z) = F'(z)$

Define analytic function

A function $F(z)$ is differentiable at $z = z_0$ and the neighbourhood points of z_0 is called analytic function

A function $F(z)$ is analytic in a domain D (F is differentiable at all points in D)

then $F(z)$ is analytic in D

$F(z)$ is analytic at $z = z_0, z_1, z_2, \ldots z_n$ in D

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} \text{ exists}$$

$$F'(z_1) = \lim_{z \to z_1} \frac{F(z) - F(z_1)}{z - z_1} \text{ exists}$$

$$F'(z_2) = \lim_{z \to z_2} \frac{F(z) - F(z_2)}{z - z_2} \text{ exists}$$

At $z = z_n$ $F'(z_n) = \lim_{z \to z_n} \frac{F(z) - F(z_n)}{z - z_n}$ exists

Problems:

① Test the function $f(z)$ is Analytic (or) not

$$\text{CR eqn} \begin{cases} u_x = v_y \\ u_y = -v_x \Rightarrow v_x = -u_y \end{cases}$$

Sol:

$$z = x+iy$$

$$f(z) = u+iv$$

$$f(z) = z = x+iy = u+iv$$

$$u = x \Rightarrow u_x = 1 \quad \Bigg| \quad v = y$$
$$v_y = 1$$

$$u_y = 0 \quad | \quad v_x = 0$$

$$u_x = v_y = 1 \quad | \quad u_y = -v_x = 0$$

$\therefore$ $f(z)$ is analytic function.

② Test whether the given function $f(z) = z^2$

Sol:

$$(z)^2 = (x+iy)^2$$

$$f(z) = u+iv$$

$$z = f(z) = (x+iy)^2 = u+iv$$

$$\Rightarrow x^2 - y^2 + i2xy = u+iv$$

$$(x^2-y^2) + i(2xy) = u+iv$$

$$u = x^2 - y^2 \quad | \quad v = 2xy$$

$$u_x = 2x \quad \Bigg| \quad v_x = 2y$$

$$u_y = -2y \quad \Bigg| \quad v_y = 2x$$

CR eqn $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$u_x = v_y = 2x$

$u_y = -v_x \Rightarrow -2y = -2y$

$\therefore$ f(z) is an analytic function

③ $f(z) = e^z \longrightarrow$ ①

Sol:

$z = x+iy$

$F(z) = u+iv = e^z$

$e^z = u+iv$

$e^{x+iy} = u+iv$

$e^x \cdot e^{iy} = u+iv$

$e^x(\cos y + i\sin y) = u+iv$

$e^x \cos y + i e^x \sin y = u+iv$

$u = e^x \cos y$ | $v = e^x \sin y$

$u_x = e^x \cos y$ | $v_x = e^x \sin y$

$u_y = -e^x \sin y$ | $v_y = e^x \cos y$

CR eqn $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

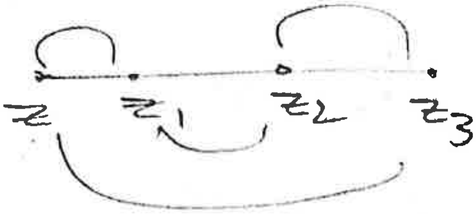
$u_x = v_y = e^x \cos y$

$u_y = -v_x \Rightarrow -e^x \sin y = -e^x \sin y$

$\therefore$ F(z) is an analytic function.

(*) Find the bilinear transformation that maps the point $\infty, i, 0$. Find the point $0, i, \infty$ of the w-plane.

Sol:



$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)}$$

$$\frac{\cancel{z_1}\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\cancel{z_1}\left(1-\frac{z_2}{z_1}\right)} = \cancel{\frac{w_1\left(\frac{w}{w_1}-1\right)(w_2-w_3)}{(w-w_3)\left(1-\frac{w_2}{w_1}\right)}}$$

$$= \frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_1-w_2)}$$

$$\Rightarrow \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{(z-0)\left(1-\frac{i}{\infty}\right)} = \frac{(w-0)\left(\frac{i}{\infty}-1\right)}{\left(\frac{w}{\infty}-1\right)(0-i)}$$

$$\frac{(0-1)(i)}{z(1-0)} = \frac{w(-1)}{(-1)(-i)}$$

$$\frac{-i}{z} = \frac{-w}{i} \Rightarrow w = \frac{i^2}{z} = \frac{-1}{z}$$

Verification:

$z=\infty,\ w = -\frac{1}{\infty} = 0$

$z = i,\ w = -\frac{1}{i}\times\frac{i}{i} = \frac{i}{-(i)^2} = \frac{i}{1} = i$

$z = 0,\ w = \frac{-1}{0} = \infty$

*) Find the Bilinear that maps the point $\infty, i, 0$ in z-plane. Find the point $0, -i, \infty$ of the w-plane.

Sol:

$z \quad z_1 \quad z_2 \quad z_3$

$z_1 = \infty, \quad z_2 = i \quad z_3 = 0$

$w_1 = 0, \quad \cancel{w_2 = i} \quad w_3 = \infty$

$w_2 = -i$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)}$$

$$\frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\cancel{z_1}\left(1-\frac{z_2}{z_1}\right)} = \frac{(w-w_1)\cancel{w_3}\left(\frac{w_2}{w_3}-1\right)}{\cancel{w_3}\left(\frac{w}{w_3}-1\right)(w_1-w_2)}$$

$$\frac{\left(\frac{z}{\infty}-1\right)(i-0)}{(z-0)\left(1-\frac{i}{\infty}\right)} = \frac{(w-0)\left(\frac{(-i)}{\infty}-1\right)}{\left(\frac{w}{\infty}-1\right)(0-(-i))}$$

$$\frac{(0-1)i}{z(1-0)} = \frac{w(0-1)}{(0-1)i} \Rightarrow \frac{-i}{z} = \frac{-w}{-i}$$

$$w = \frac{-i^2}{z} = \frac{-(-1)}{z} = \frac{1}{z}$$

Verification

$z_1 = \infty \Rightarrow w = \frac{1}{\infty} = 0$

$z_2 = i \Rightarrow w = \frac{1}{i} \times \frac{-i}{-i} \Rightarrow w = \frac{-i}{-i^2} = \frac{-i}{+1} = -i$

$z_3 = 0 \Rightarrow w = \frac{1}{0} = \infty$

(*) Find the Bilinear Transformation that maps the point $-1, 0, 1$ on to the points $-1, -i, 1$

Sol:- $z = -1, 0, 1$

$w = -1, -i, 1$

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_1 - z_2)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_1 - w_2)}$$

$$\frac{(z+1)(0-1)}{(z-1)(-1-0)} = \frac{(w+1)(-i-1)}{(w-1)(-1+i)}$$

$$\frac{(z+1)(-1)}{(z-1)(-1)} = \frac{(w+1)(-i-1)}{(w-1)(-1+i)}$$

$$\frac{z+1}{z-1} = \frac{\cancel{-}(w+1)(i+1)}{\cancel{-}(w-1)(1-i)}$$

$$\frac{z+1}{z-1} = \frac{(w+1)(1+i)}{(w-1)(1-i)}$$

$$\frac{z+1}{z-1} = \frac{w+1+iw+i}{w-1-iw+i}$$

$$(z+1)(w-1-iw+i) = (z-1)(w+1+iw+i)$$

$$\cancel{zw} + w - \cancel{z} - 1 - izw - \cancel{iw} + \cancel{iz} + i = \cancel{wz} - w + \cancel{z} - 1$$
$$+ izw - \cancel{iw} + \cancel{iz} - i$$

$$w - z - izw + i + w - z - izw + i = 0$$

$$2w - 2z - 2izw + 2i = 0$$

$$2w(1 - iz) + 2(-z + i) = 0$$

$$2[w(1 - iz) + i - z] = 0$$

$$w(1-iz) = z-i$$

$$\boxed{w = \frac{z-i}{1-iz}}$$

Verification:

$$z_1 = -1 \Rightarrow w = \frac{-1-i}{1+i} \times \frac{1-i}{1-i} = \frac{-2}{1+i^2} = -\frac{2}{2} = -1$$

$$z_2 = 0 \Rightarrow w = \frac{0-i}{1-0} = -i$$

$$z_3 = 1 \Rightarrow w = \frac{1-i}{1-i} = 1$$

* Find the fixed point of the Transformation $w = \frac{6z-9}{z}$

Sol:

$$w = f(z) = \frac{6z-9}{z}$$

$$z = \frac{6z-9}{z}$$

$$z^2 = 6z-9$$

$$z^2 - 6z + 9 = 0$$

$$(z-3)(z-3) = 0$$

$z-3=0$	$z-3=0$
$\boxed{z=3}$	$\boxed{z=3}$

$z = 3, 3$ Fixed points.

(*) Find the Fixed Point of the Transformation

$$\omega = \frac{2z+6}{z+7}$$

Sol:- $\omega = f(z) = \dfrac{2z+6}{z+7}$

$$z = \frac{2z+6}{z+7}$$

$$z(z+7) = 2z+6$$

$$z^2 + 7z - 2z - 6 = 0$$

$$z^2 + 5z - 6 = 0$$

$$(z-1)(z+6) = 0$$

$$\boxed{z=1} \;\; | \;\; \boxed{z=-6}$$

(*) Find the Fixed point of Transformation

$$\omega = \frac{3-z}{1+z}$$

Sol:- $\omega = f(z) = \dfrac{3-z}{1+z}$

$$z = \frac{3-z}{1+z}$$

$$z + z^2 = 3 - z$$

$$z^2 + 2z - 3 = 0$$

$$z^2 + z + 3z - 3 = 0$$

$$(z-1)(z+3) = 0$$

$$| \;\; \boxed{z=-3}$$

Critical points:

(*) Find the Critical point of the Function

$f(x) = \dfrac{x+1}{x^2+1}$, $\int \frac{u}{v} dx = \dfrac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$

Sol: $f'(x) = 0$

$$f'(x) = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2}$$

$$0 = \frac{x^2+1-2x^2+2x}{x^4+1^4}$$

$$0 = \frac{-x^2+2x+1}{x^4+1}$$

$$x^2+2x-1=0$$

$$a=1,\ b=2,\ c=-1$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-2\pm\sqrt{4+4}}{2} = \frac{-2\pm\sqrt{8}}{2}$$

$$= \frac{-2\pm2\sqrt{2}}{2} = -1\pm\sqrt{2}$$

(*) Find the Critical point of the function

$f(x) = \dfrac{x^2-1}{x^2+1}$

Sol:

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$0 = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$

$\cancel{2x^3}+2x-\cancel{2x^3}+2x=0$

$4x=0$

$4\neq 0 \Rightarrow \boxed{x=0}$

*Conformal Mapping: (Image to Image Transformation)

(*) Find the image of the Region bounded by $x=0, y=0, x=1, y=2$. under the transformation $w=z+2-i$

Sol:- $w=f(z) \rightarrow (1)$

$z=x+iy \rightarrow (2)$

$f(z)=u+iv \rightarrow (3)$

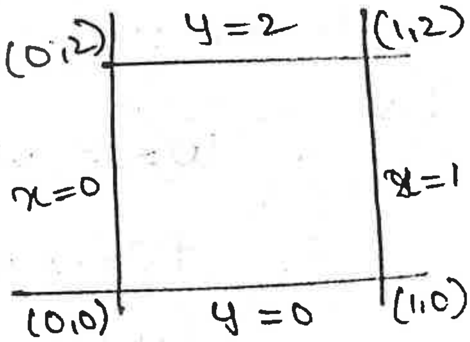
$w=f(z)=z+2-i$

$u+iv=x+iy+2-i$

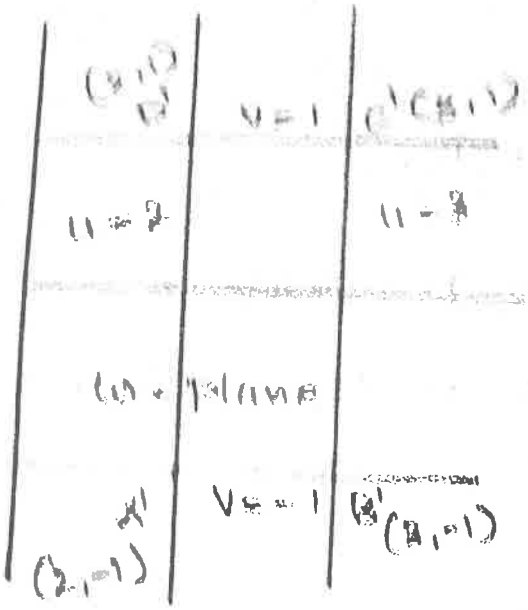
Compare real (or) Imaginary Part

$u+iv=(x+2)+i(y-1)$

$u=x+2 \quad , \quad v=y-1$



Points	$u=x+2$	$V=y-1$	(u,v)
A(0,0)	$u=0+2$ $u=2$	$V=-1$	$(2,-1)$
B(0,1)	$u=3$	$V=-1$	$(3,-1)$
C(1,2)	$u=3$	$V=1$	$(3,1)$
D(0,2)	$u=2$	$V=1$	$(2,1)$



Conclusion:

A rectangle bounded by 4 straight lines $x=0, y=0$ & $x=1, y=2$ in a z-plane is transfor-
-med into a rectangle $u=2, u=3$ & $v=-1, v=1$ in w-plane under the mapping $z+2-i$.

(*) Find the image of the rectangular region bounded by the region $x=0, y=0, x=1, y=2$ under the transformation $w=(1+i)z$

Sol:

$$w = f(z)$$

$$f(z) = u+iv$$

$$z = x+iy$$

$$w = (1+i)z$$

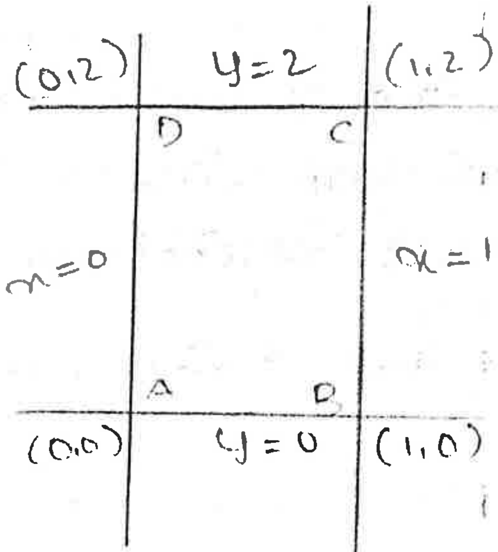
$$w = (1+i)(x+iy)$$

$$w = x+xi+iy+(-1)y$$

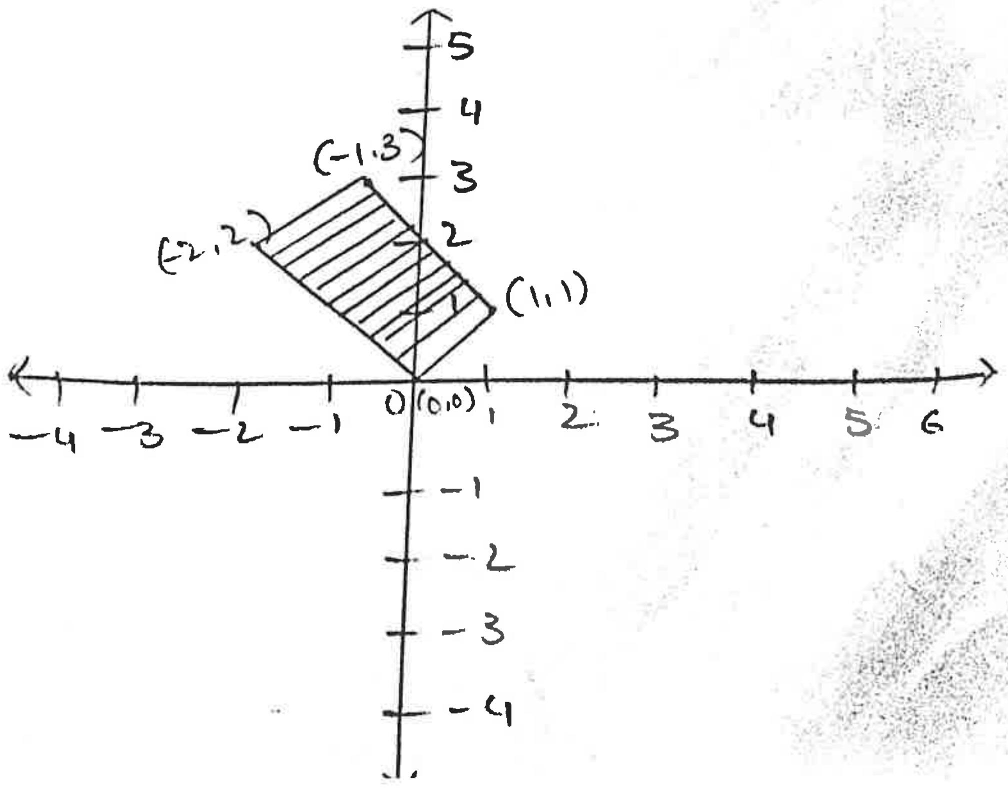
$$w = \cancel{x(1+i)}\ i(x+y)+x-y$$

$$w = x-y+i(x+y)$$

$u = x - y \quad , \quad v = x + y$



Points	$u = x - y$	$v = x + y$	(u,v)
A(0,0)	$u = 0$	$v = 0$	(0,0)
B(1,0)	$u = 1$	$v = 1$	(1,1)
C(1,2)	$u = -1$	$v = 3$	(-1,3)
D(0,2)	$u = -2$	$v = 2$	(-2,2)



(*) ~~Draw~~ Find the image of the circle $|z|=2$ under the transformation $w=3z$

Sol:

$$w = 3z \rightarrow (1)$$

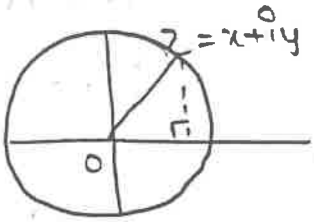
$$w = f(z) = u+iv \rightarrow (2) \quad z = x+iy$$

$$u+iv = 3(x+iy)$$

$$u+iv = 3x+i3y$$

$u = 3x$ | $v = 3y$

$\boxed{x = \frac{u}{3}}$ | $\boxed{y = \frac{v}{3}}$



$x^2+y^2=r^2$

$|z| = \sqrt{x^2+y^2}$

$$|z| = 2$$

$$|x+iy| = 2$$

$$\sqrt{x^2+y^2} = 2 \rightarrow (3)$$

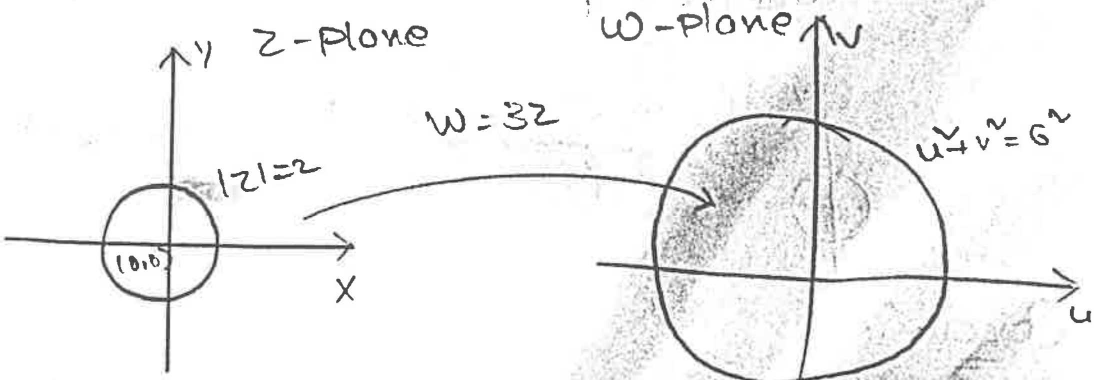
$$x^2+y^2 = 4$$

$$\frac{u^2}{9} + \frac{v^2}{9} = 4$$

$$\frac{u^2+v^2}{9} = 4$$

$$u^2+v^2 = 36$$

$$\boxed{u^2+v^2 = 6^2} \rightarrow (4)$$



Conclusion:

A circle $|z|=2$ with the centre $(0,0)$ & radius 2 unit in a z-plane is transformed into a circle in w-plane with centre $(0,0)$ and

$$= 4abc\left[\frac{2a^3}{3} + \frac{2ab^2}{3} + \frac{2ac^2}{3}\right]$$

$$= 4abc\,(2a)\left[\frac{a^2}{3} + \frac{b^2}{3} + \frac{c^2}{3}\right]$$

$$= \frac{8a^2bc}{3}\,(a^2+b^2+c^2) \text{ Cubic units}$$

(iii) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz}{\sqrt{a^2-x^2-y^2-z^2}}\right) dy\right] dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}}\,dz\right) dy\right] dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \left(\sin^{-1}\left(\frac{z}{\sqrt{a^2-x^2-y^2}}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\right)\right] dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sin^{-1}(1)\,dy\right] dx$$

$$= \frac{\pi}{2}\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy\right] dx$$

$$= \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \cancel{\frac{\pi}{2}\left[a^2\int_0^a dx - \int_0^a x^2 dx\right]} = \cancel{\frac{\pi}{2}a^2(a) - \left[\frac{x^3}{3}\right]_0^a} = \cancel{\frac{\pi a^3}{2} - \frac{\pi a^3}{3}}$$

$$= \frac{\pi}{2}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

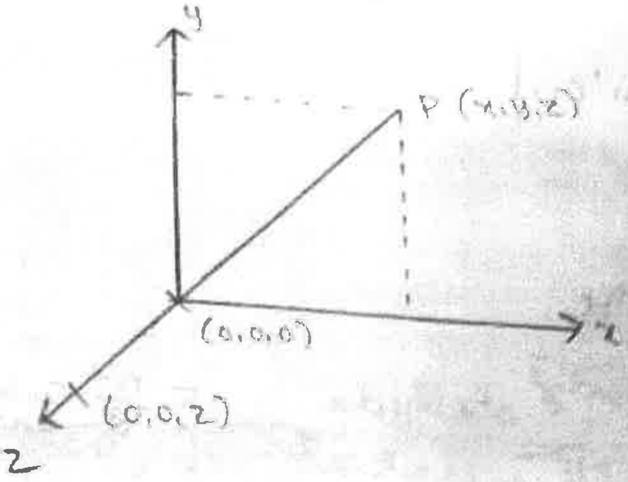
$$= \frac{\pi}{2}\left[0+\frac{a^2}{2}\sin^{-1}(1)\right]$$

$$= \frac{\pi}{2}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi^2 a^2}{8} \text{ cubic units}$$

Unit - II

Vector Calculus



Positive Vector ; $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

(3D coordinate

2D coordinate : $\vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$

Formulae:

$$\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

$$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\vec{a} + \vec{b} = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j} + (z_1 + z_2)\vec{k}$$

$$\vec{a} - \vec{b} = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j} + (z_1 - z_2)\vec{k}$$

Multiply

Let $\vec{i}, \vec{j}, \vec{k}$ are coordinate

$\vec{i}\cdot\vec{i} = 1$	$\vec{i}\cdot\vec{j} = 0$
$\vec{j}\cdot\vec{j} = 1$	$\vec{j}\cdot\vec{k} = 0$
$\vec{k}\cdot\vec{k} = 1$	$\vec{k}\cdot\vec{i} = 0$

Dot Product:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta\,\hat{n}$$

$\hat{n}$ - unit outward normal vector ($\hat{n}=1$)

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{i}\cdot\vec{i} = |\vec{i}||\vec{i}|\cos\theta$$

$$\vec{j}\cdot\vec{j} = |\vec{j}||\vec{j}|\cos\theta$$

$$\vec{i}\cdot\vec{i}=1,\quad \vec{j}\cdot\vec{j}=1,\quad \vec{k}\cdot\vec{k}=1$$

$$\vec{i}\cdot\vec{j} = |\vec{i}|.|\vec{j}|\cos\theta$$
$$= |\vec{i}||\vec{j}|\cos 90^\circ$$
$$= 0$$

$\vec{i}\,.\,\vec{j}$ are $\perp^r$ to each other

$$\vec{j}\cdot\vec{k}=0,\quad \vec{k}\cdot\vec{i}=0$$

Gradient:

Let $\phi(x,y,z)$ be a Scalar point function then

$$\nabla\phi = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\phi(x,y,z)$$

$$\nabla\phi = \left(\vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}\right)$$

Where, ∇ is a Gradient operator.

$$\boxed{\nabla = \vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}}$$

$\nabla \to$ Vector differential operator

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$$

(1^{st} order +ve Derivatives)

$$y = x^2$$

$$\frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2 \Rightarrow \frac{d^3y}{dx^3} = 0$$

(ordinary Derivati

$\nabla\phi$ is a Vector point function.

Unit Normal Vector:-

~~The~~ Unit normal Vector to the ~~given~~ Surface $\phi(x,y,z)$ is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Directional Derivative:

The directional derivative for the Surface $\phi(x,y,z)$ in the direction of the Vector $\vec{a}$ is giv by Directional derivative $= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$\boxed{D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}}$$

Maximum Directional Derivative :-

The Max. Directional derivative to the Surface is given by

$$\text{Max. D.D} = |\nabla\phi|$$

Angle b/w Surfaces :-

Let $f(x,y,z)$ is a be any two Surfaces &
$g(x,y,z)$ is a be any two Surfaces then
the angle b/w two Surfaces is given by

$$\boxed{\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}}$$

Note:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta\,\hat{n}$$

$\hat{n} \to$ Unit normal Vector

$\hat{n} = 1$, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

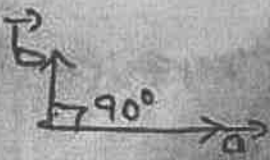
$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}} \quad \vec{a} = \nabla f, \quad \vec{b} = \nabla g$$

$\vec{a} \times \vec{b}$ both Parallel

$\theta = 0^\circ$

$$\cos 0^\circ = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow 1 = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$$

Perpendicular $\vec{a} \perp \vec{b}$



$\theta = 90^\circ$

$$\cos 90^\circ = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \vec{a}\cdot\vec{b} = 0 \text{ (or) } \nabla f \cdot \nabla g = 0$$

$f(x,y,z)$ & $g(x,y,z)$ two Surfaces are $\perp^{er}$

(*) $\phi(x,y,z) = x^2+y^2+z^2-a^2$ (Sphere $= x^2+y^2+z^2=a^2$)

Find $D\phi$?

Sol:

$$D = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right) \rightarrow \text{vector. funct.}$$

$$\phi(x,y,z) = x^2+y^2+z^2-a^2 \rightarrow \text{Sphere Surface}$$

$$D\phi = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\phi$$

$$= \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{d\phi}{dy} + \vec{k}\frac{d\phi}{dz}$$

$$= \vec{i}\frac{d}{dx}(x^2+y^2+z^2-a^2) + \vec{j}\frac{d}{dy}(x^2+y^2+z^2-a^2) + \vec{k}\frac{d}{dz}(x^2+y^2+z^2-a^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\boxed{D\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{j}}$$

(*) $D\phi$ at $(1,1,1)$

$$D\phi = 2(1)\vec{j} + 2(1)\vec{j} + 2(1)\vec{k}$$

$$\boxed{D\phi = 2\vec{i} + 2\vec{j} + 2\vec{k}}$$

(*) $\phi(x,y,z) = xyz$ $(1,-1,0)$

(*) $\phi(x,y,z) = x^2y + y^2z + z^2x$ $(1,1,1)$

(*) $\phi(x,y,z) = x^3+y^3+z^3$ $(1,1,1)$

$\cancel{(1) \ \phi(x,y,z) = xyz \ (1,-1,0)}$

$\cancel{D\phi = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\phi}$

1) $\phi(x,y,z) = xyz \quad (1,-1,0)$

Sol:

$$D\phi = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\phi$$

$$D\phi = \vec{i}\frac{d}{dx}(xyz) + \vec{j}\frac{d}{dy}(xyz) + \vec{k}\frac{d}{dz}(xyz)$$

$$= \vec{i}(1)(yz) + \vec{j}(x)(1)(z) + \vec{k}(x)(y)(1)$$

$$D\phi = (yz)\vec{i} + (xz)\vec{j} + (xy)\vec{k}$$

at $(1,-1,0)$

$$D\phi = (-1)(0)\vec{i} + (1)(0)\vec{j} + (1)(-1)\vec{k}$$

$$= 0 + 0 - \vec{k}$$

$$\boxed{D\phi = -\vec{k}}$$

2

Sol: $\phi(x,y,z) = x^2y + y^2z + z^2x \quad (1,1,1)$

$$D\phi = \vec{i}\frac{d}{dx}(x^2y+y^2z+z^2x) + \vec{j}\frac{d}{dy}(x^2y+y^2z+z^2x)$$
$$+ \vec{k}\frac{d}{dz}(x^2y+y^2z+z^2x)$$

$$D\phi = (2xy+z^2)\vec{i} + (x^2+2yz)\vec{j} + (y^2+2zx)\vec{k}$$

at $(1,1,1)$

$$D\phi = (2(1)+(1)^2)\vec{i} + ((1)^2+2(1))\vec{j} + ((1)^2+2(1))\vec{k}$$

$$= 3\vec{i} + 3\vec{j} + 3\vec{k}$$

③ $\phi(x,y,z) = x^3+y^3+z^3 \quad (1,1,1)$

Sol:-

$$D\phi = \vec{i}\frac{d}{dx}(x^3+y^3+z^3) + \vec{j}\frac{d}{dy}(x^3+y^3+z^3) + \vec{k}\frac{d}{dy}(x^3+y^3+z^3)$$

$$= \vec{i}(3x^2) + \vec{j}(3y^2) + \vec{k}(3z^2)$$

$$D\phi = (3x^2)\vec{i} + (3y^2)\vec{j} + (3z^2)\vec{k}$$

at $(1,1,1)$

$$D\phi = 3(1)^2\vec{i} + 3(1)^2\vec{j} + 3(1)^2\vec{k}$$

$$= 3\vec{i} + 3\vec{j} + 3\vec{k}$$

(4) $D\phi$ (or) grad ϕ at $(1,0,0)$ if $\phi = x^2+y-z-1$

$$D\phi = \vec{i}\frac{d}{dx}(x^2+y-z-1) + \vec{j}\frac{d}{dy}(x^2+y-z-1)$$

$$+ \vec{k}\frac{d}{dz}(x^2+y-z-1)$$

$$= \vec{i}\cancel{\frac{d}{dx}}(2x) + \vec{j}(1) + \vec{k}(-1)$$

$$= 2x\vec{i} + \vec{j} - \vec{k}$$

at $(1,0,0)$

$$= 2(1)\vec{i} + \vec{j} + (-\vec{k})$$

$$= 2\vec{i} + \vec{j} - \vec{k}$$

⑤ $\phi = (x y z)$ at $(1,1,1)$ Find $D\phi$

$$D\phi = \vec{i}\frac{d}{dx}(xyz) + \vec{j}\frac{d}{dy}(xyz) + \vec{k}\frac{d}{dz}(xyz)$$

$$= \cancel{i\frac{d}{dx}yz} = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$= (yz)\vec{i} + (xz)\vec{j} + (xy)\vec{k}$$

at $(1,1,1)$

$$= (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$$

$$D\phi = \vec{i} + \vec{j} + \vec{k}$$

⑥ Find $D\phi$ if $\phi = \log(x^2+y^2+z^2)$ at

$$D\phi = \vec{i}\frac{d}{dx}(\log(x^2+y^2+z^2)) + \vec{j}\frac{d}{dy}(\log(x^2+y^2+z^2))$$

$$+ \vec{k}\frac{d}{dz}(\log(x^2+y^2+z^2))$$

$$= \vec{i}\,\cancel{\log}\frac{1}{x^2+y^2+z^2}\cdot 2x + \vec{j}\frac{1}{x^2+y^2+z^2}2y + \vec{k}\frac{1}{x^2+y^2+z}2z$$

$$= \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z}\vec{k}$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

at $(1,1,1)$

$$D\phi = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k})$$

(7) $\nabla \vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)$$

$$\nabla\vec{r} = \vec{i}\frac{d}{dx}(x\vec{i} + y\vec{j} + z\vec{k}) + \vec{j}\frac{d}{dy}(x\vec{i} + y\vec{j} + z\vec{k}) + \vec{k}\frac{d}{dz}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla\vec{r} = y(\vec{i}\cdot\vec{j}) + z(\vec{i}\cdot\vec{k}) + 1\frac{d}{dx}(x)$$

$$+ \vec{j}\cdot\vec{i}(x) + z(\vec{j}\cdot\vec{k}) + (1)\frac{d}{dy}(y)$$

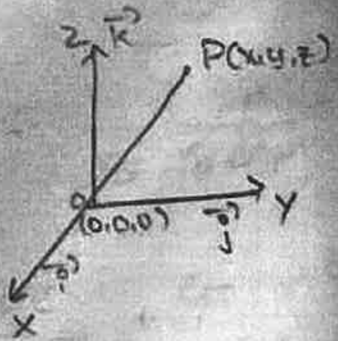
$$+ 0 + 0 + \frac{d}{dz}(z)$$

$$\nabla\vec{r} = 1+1+1 = 3$$

(8) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ Find $\nabla r = \frac{\vec{r}}{r}$

Sol:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$



$$\overrightarrow{OP} = d^2 = x^2 + y^2 + z^2$$

$$r^2 = x^2 + y^2 + z^2$$

$$\vec{r}\cdot\vec{r} = r^2 = (x\vec{i} + y\vec{j} + z\vec{k})\cdot(x\vec{i} + y\vec{j} + z\vec{k})$$

$$r^2 = x^2 + y^2 + z^2 \rightarrow (1)$$

$r(x, y, z)$

$$\frac{dr}{dx}, \frac{dr}{dy}, \frac{dr}{dz}$$

(1) Partially diff w.r.to x

$$2r\frac{dr}{dx} = 2x$$

$$\boxed{\frac{dr}{dx} = \frac{x}{r}}$$

(2) Partially diff. w.r.to 'y'

$$2r\frac{dr}{dy} = 2y$$

$$\boxed{\frac{dr}{dy} = \frac{y}{r}}$$

(3) Partially diff. w.r.to 'z'

$$2r\frac{dr}{dz} = 2z$$

$$\boxed{\frac{dr}{dz} = \frac{z}{r}}$$

$$\nabla r = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)r$$

$$= \vec{i}\frac{dr}{dx} + \vec{j}\frac{dr}{dy} + \vec{k}\frac{dr}{dz}$$

$$= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

$$= \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\nabla r = \frac{1}{r}(\vec{r}) = \frac{\vec{r}}{r}$$ //

(9) $\nabla r^n = n r^{n-2}[\vec{r}]$

Sol:

$$\nabla r^n = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)r^n$$

$$= \vec{i}\frac{d}{dx}(r^n) + \vec{j}\frac{d}{dy}(r^n) + \vec{k}\frac{d}{dz}(r^n)$$

$$= (n r^{n-1})\vec{i}\frac{dr}{dx} + (n r^{n-1})\vec{j}\frac{dr}{dy} + (n r^{n-1})\vec{k}\frac{dr}{dz}$$

$$= n r^{n-1}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right]$$

$= \frac{n r^{n-1}}{r} [x\vec{i} + y\vec{j} + z\vec{k}]$

$= n r^{n-2} \vec{r}$ //

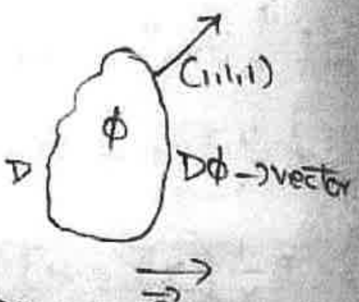
(10) Find Dir. Derivative $\phi = xyz$ at $(1,1,1)$ in the Dir. of $\vec{a} = \vec{i}+\vec{j}+\vec{k}$

Sol:- $D.D = D\phi \, \frac{\vec{a}}{|\vec{a}|}$

$$D\phi = \vec{i}\frac{d}{dx}(xyz) + \vec{j}\frac{d}{dy}(xyz) + \vec{k}\frac{d}{dz}(xyz)$$

$$= (yz)\vec{i} + (xz)\vec{j} + (xy)\vec{k}$$

$$D\phi = \vec{i} + \vec{j} + \vec{k}$$



$$D.D = D\phi \, \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i}+\vec{j}+\vec{k}) \cdot \frac{(\vec{i}+\vec{j}+\vec{k})}{|\vec{i}+\vec{j}+\vec{k}|}$$

$$= \frac{1}{|\vec{i}+\vec{j}+\vec{k}|}$$

$$= \frac{1+1+1}{\sqrt{(1)^2+(1)^2+(1)^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

(11) $\phi = xy+yz+zx$ at $(1,2,0)$ $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$

$$D\phi = \vec{i}\frac{d}{dx}(xy+yz+zx) + \vec{j}\frac{d}{dy}(xy+yz+zx)$$
$$+ \vec{k}\frac{d}{dz}(xy+yz+zx)$$

$$= \vec{i}\cancel{\frac{d}{dx}}(y+z) + \vec{j}(x+z) + \vec{k}(x+y)$$

$= (2+0)\vec{i} + (1+0)\vec{j} + (1+2)\vec{k}$

$\nabla\phi = 2\vec{i} + \vec{j} + 3\vec{k}$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i} + \vec{j} + 3\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{|\vec{i} + 2\vec{j} + 2\vec{k}|}$$

$$= \frac{2+2+6}{\sqrt{(1)^2+(2)^2+(2)^2}} = \frac{10}{\sqrt{9}} = \frac{10}{3}$$

Unit normal Vector $= \hat{n} = \dfrac{\nabla\phi \to \text{vector}}{|\nabla\phi| \to \text{scalar}}$ } $\hat{n}$ is vector

length is 1 unit

(12) $\phi = x^2 - y^2 + z = 2$ at $(1, -1, 2)$

Sol:- $\nabla\phi = \vec{i}\dfrac{d}{dx}(x^2-y^2+z-2) + \vec{j}\dfrac{d}{dy}(x^2-y^2+z-2)$

$+ \vec{k}\dfrac{d}{dz}(x^2-y^2+z-2)$

$\nabla\phi = \vec{i}(2x) + \vec{j}(-2y) + \vec{k}$

$= 2x\vec{i} - 2y\vec{j} + \vec{k}$

$\nabla\phi = 2\vec{i} + 2\vec{j} + \vec{k}$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{2^2+2^2+1^2}}$$

$$= \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{9}}$$

$$\boxed{\hat{n} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}}$$

(13) Find the unit normal vector to the surface $\phi = x^3 - xyz + z^3 - 1$ at the point $(1,1,1)$

Sol: $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) x^3 - xyz + z^3 - 1$$

$$= \vec{i}\frac{\partial}{\partial x}(x^3 - xyz + z^3 - 1) + \vec{j}\frac{\partial}{\partial y}(x^3 - xyz + z^3 - 1) + \vec{k}\frac{\partial}{\partial z}(x^3 - xyz + z^3 - 1)$$

$$= \vec{i}(3x^2 - yz) + \vec{j}(-xz) + \vec{k}(-xy + 3z^2)$$

$$= (3x^2 - yz)\vec{i} - \vec{j}(xz) + (3z^2 - xy)\vec{k}$$

at $(1,1,1)$

$$= (3-1)\vec{i} + (-1)\vec{j} + (3-1)\vec{k}$$

$$\boxed{\nabla\phi = 2\vec{i} - \vec{j} + 2\vec{k}}$$

$$|\nabla\phi| = \sqrt{(2)^2 + (-1)^2 + (2)^2}$$

$$= \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}$$

$$|\hat{n}| = \left|\frac{\nabla\phi}{|\nabla\phi|}\right| = \frac{|\nabla\phi|}{|\nabla\phi|} = 1$$

$$|\hat{n}| = \left|\frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}\right| = \frac{3}{3} = 1$$

(*) Divergence of a vector point function $\vec{F}$, ie $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then the divergence of $\vec{F}$ is $\nabla\vec{F} = \dfrac{\partial}{\partial x}F_1 + \dfrac{\partial}{\partial y}F_2 + \dfrac{\partial}{\partial z}F_3$

(i) $\nabla\vec{F}$ is a scalar quantity

(ii) If $\nabla\cdot\vec{F} = 0$, then $\vec{F}$ is said to be Solenoidal

(iii) Curl of a Vector point function

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ then

$$\nabla\times\vec{F} = \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

(i) Curl $\vec{F}$ is a Vector Quantity.

(ii) If Curl $\vec{F}=0$ then $\vec{F}$ is said to be irrotational.

Solenoidal and Irrotational

1) Find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$ of the Vector points functions. Find $\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$ at the point $(1,-1,1)$

Sol:

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(xz^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2z + 2y(\cancel{8}4z^3)$$

$$\nabla\cdot\vec{F} = z^3 - 2x^2z + 8yz^3$$

at $(1,-1,1)$

$$\nabla\cdot\vec{F} = (1)^3 - 2(1)^2(1) + 8(-1)(1)$$

$$= 1 - 2 - 8 = -9$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2x^3yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \vec{i}\left[2z^4 + 2x^2y\right] - \vec{j}\left[0 - 3xz^2\right] + \vec{k}\left[-4xyz - 0\right]$$

Point $= (1,-1,1)$

$$\nabla\times\vec{F} = \vec{i}\left[2(1)^4 + 2(1)^2(-1)\right] - \vec{j}\left[-3(1)(1)^2\right] + \vec{k}\left[-4(1)(-1)(1)\right]$$

$$= \vec{i}\,[2-2] - \vec{j}\,[-3] + \vec{k}\,[4]$$

$$\nabla\times\vec{F} = 3\vec{j} + 4\vec{k}$$

② If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$. Find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$.

Sol:-

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla\cdot\vec{F} = 2x + 2y + 2z$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\nabla\times\vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0)$$

$$\boxed{\nabla\times\vec{F} = \vec{0}}$$

③ Find the Value of 'a' Show that the Vector. $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is Solenoidal.

Sol:- Solenoidal $\nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left((x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$\cancel{= 1+0+1+0 \Rightarrow 1+0+1+0+1+a = 0}$$

$$\Rightarrow 1+1+a = 0$$

$$\boxed{a = -2}$$

④ Show that the Vector $\vec{F} = (3y^4z^2)\vec{i} + (4x^3z^2)\vec{j} - 3x^2y^2)\vec{k}$ is a Solenoidal.

Sol:- $\nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2) = 0$$

$$0+0+0 = 0 \Rightarrow \boxed{0=0}$$

Hence Proved

⑤ Show that $\vec{F} = (y^2+2xz^2)\vec{i} + (2xy-z)\vec{j} + (2x^2z-y+2z)\vec{k}$ is irrotational.

Sol:- Irrotational, $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}\left[\frac{\partial}{\partial y}(2x^2z-y+2z) - \frac{\partial}{\partial z}(2xy-z)\right] - \vec{j}\left[\frac{\partial}{\partial z}(y^2+2xz^2) - \frac{\partial}{\partial x}(2x^2z-y+2z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2xy-z) - \frac{\partial}{\partial y}(y^2+2xz^2)\right] = 0$$

$$\Rightarrow \vec{i}\left[(-1) - (-1)\right] - \vec{j}\left[(4xz) - (4xz)\right] + \vec{k}\left[(2y) - (2y)\right] = 0$$

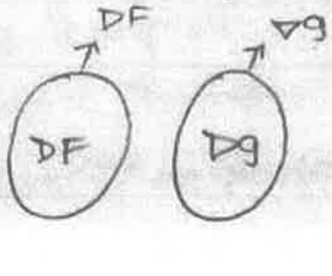
$$\Rightarrow \vec{i}(0) - \vec{j}(0) + \vec{k}(0) \Rightarrow \boxed{0=0}$$

Hence proved irrotational

~~⑥ Find a unit volume normal vector to the Surface~~

~~$\phi = x^3 - xyz$~~

Angle b/w two Surfaces:



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \rightarrow \frac{\text{Scalar}}{\text{Scalar}} = \text{Scalar}$$

$\cos\theta = \text{scalar}$

$\theta = \cos^{-1}(\,)$

$\nabla f \cdot \nabla g$

$$\cos\theta = \frac{\vec{f}\cdot\vec{g}}{|\vec{f}||\vec{g}|}$$

(1) Find the angle b/w Surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at $(2,-1,2)$

Sol:-

$$\cos\theta = \frac{\vec{f}\cdot\vec{g}}{|\vec{f}||\vec{g}|}$$

$$\cos\theta = \cancel{(x^2+y^2+z^2-9)}($$

Let $f(x,y,z) = x^2+y^2+z^2-9$

$g(x,y,z) = x^2+y^2-z-3=0$

$$\nabla f = \left(\vec{i}\frac{d}{dx}(x^2+y^2+z^2-9) + \vec{j}\frac{d}{dy}(x^2+y^2+z^2-9) + \vec{k}\frac{d}{dz}(x^2+y^2+z^2-9)\right)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$= 2x(\vec{i}) + (2y)\vec{j} + 2z(\vec{k})$$

$(2,-1,2)$ $= 2(2\vec{i}) + 2(-1)\vec{j} + 2(2)\vec{k}$

$$\boxed{\nabla f = 4\vec{i} - 2\vec{j} + 4\vec{k}}$$

$$\nabla g = \left(\vec{i}\frac{d}{dx}(x^2+y^2-z-3) + \vec{j}\frac{d}{dy}(x^2+y^2-z-3) + \vec{k}\frac{d}{dz}(x^2+y^2-z-3)\right)$$

$\nabla g = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$

$= 2x(\vec{i}) + 2y(\vec{j}) - \vec{k}$

$(2, -1, 2)$

$$\boxed{\nabla g = 4\vec{i} - 2\vec{j} - \vec{k}}$$

$$\cos\theta = \frac{\vec{f} \cdot \vec{g}}{|\vec{f}||\vec{g}|}$$

$|\nabla f| = \sqrt{4^2 + (-2)^2 + (4)^2}$

$= \sqrt{16+4+16}$

$= \sqrt{36} = 6$

$|\nabla g| = \sqrt{4^2 + (-2)^2 + (-1)^2}$

$= \sqrt{16+4+1} = \sqrt{21}$

$$\cos\theta = \frac{\nabla\vec{f} \cdot \nabla\vec{g}}{|\nabla\vec{f}||\nabla\vec{g}|}$$

$$= \frac{(4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})}{6\sqrt{21}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{\cancel{16}^{8}}{\cancel{6}_{3}\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

* Vector Integration:

① Line integral

② Green's Theorem

③ Gauss Divergence theorem

④ Stoke's Theorem

(1) Line integral:-

→ An integral evaluated along the curve C is called Line integral.

→ Let $\vec{F}$ be a vector point function acting on the path.

Then

$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)}$$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

Find $\vec{F}\cdot d\vec{r} \Rightarrow (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$$\boxed{\vec{F}\cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz}$$

Problems

① If $\vec{F} = x^2\vec{i} + y^2\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the line $y = x$ from (0,0) (1,1)

Sol:- $\vec{F} = x^2\vec{i} + y^2\vec{j}$

$\vec{r} = (x\vec{i} + y\vec{j} + 0\vec{k})$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2\vec{i} + y^2\vec{j})\cdot(dx\vec{i} + dy\vec{j})$

$= x^2 dx + y^2 dy$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (x^2 dx + y^2 dy)$$

$y = x$

$\frac{dy}{dx} = 1$

$\boxed{dy = dx}$

$$= \int_0^1 x^2 dx + x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

③ $\vec{F} = x^2\vec{i} + xy\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ $y = x$ from $(0,0)$ $(1,1)$

Sol:- $\vec{F} = x^2\vec{i} + xy\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\vec{F}\cdot d\vec{r} = (x^2\vec{i} + xy\vec{j})\cdot(dx\vec{i} + dy\vec{j})$$

$$\vec{F}\cdot d\vec{r} = x^2 dx + xy\,dy$$

$y = x$

$\boxed{dy = dx}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 x^2 dx + xy\,dy$$

$$= \int_0^1 x\,dx + x^2 dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \cancel{\frac{5}{6}} = \frac{2}{3}$$

③ $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ along the line $y = x^2$, $x = 1$, $x = 2$

Sol:- $\vec{F} = 5xy\vec{i} + 2y\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\vec{F} \cdot d\vec{r} = (5xy\vec{i} + 2y\vec{j}) \cdot (dx\vec{i} + dy\vec{j})$$

$$\vec{F} \cdot d\vec{r} = 5xy\,dx + 2y\,dy$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\boxed{dy = 2x\,dx}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C 5xy\,dx + 2y\,dy$$

$$= \int_1^2 5x(x)^2 dx + 2(x)^2 dx \cdot 2x$$

$$= \int_1^2 5x^3 dx + 4x^3 dx$$

$$= 5\left(\frac{x^4}{4}\right)_1^2 + 4\left(\frac{x^4}{4}\right)_1^2$$

$$= 5\left[\frac{16}{4} - \frac{1}{4}\right] + 4\left[\frac{16}{4} - \frac{1}{4}\right]$$

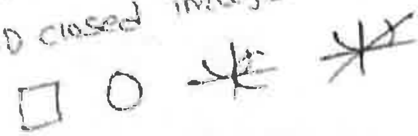
$$= \frac{15}{4} \times 5 + \frac{15}{\cancel{4}} \times \cancel{4}$$

$$= \frac{75}{4} + 15 = \frac{75+60}{4} = \frac{135}{4}$$

Green's Theorem!

(1) Relatives line integral & multiple integral

(2) 2-D closed image.



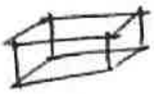
(3) $\int P\,dx + Q\,dy = \iint_R (Q_x + P_y)\,dx\,dy$

line integral   Double integral

Gauss Divergence Theorem!

① It relates double integral & triple integral

② Sphere



③ $\nabla \cdot \vec{F}$

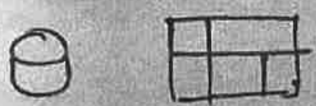
$$\iint \vec{F} \cdot \vec{n}\,ds = \iiint_V \nabla \cdot \vec{F}\,dv$$

Surface Double   Triple

Stroke's Theorem!-

① Relate line integral & Double integral

② 3D - open object



③ $\nabla \times \vec{F}$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\,ds = \oint \vec{F} \cdot d\vec{r}$$

double   Line

④ Stokes theorem is a pictular Case Green's theorem when 3D object.

Green's Theorem :-

Let $P(x,y)$, $Q(x,y)$ be any two Continuous friction with first order partial derivative exist then, $\int P dx + Q dy = \iint \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx dy$

Region R (or) xy-plane

$$\int P dx + Q dy = \iint (Q_x - P_y) dx dy \quad R \text{ on } xy\text{plane}$$

The line inregral along the boundary of the Curve 'c' is equal to the Region 'R' on the xy-Plane enclosed by 'c'.

Closed Curve :-

$$\Rightarrow \int_C P dx + Q dy = \iint_R (Q_x - P_y) dx dy \longrightarrow ①$$

$$Q = x, \quad P = -y$$

$$\frac{dQ}{dx} = 1 \qquad \frac{dP}{dy} = -1$$

$$① \Rightarrow \int_C -y dx + x dy = \iint_R [1 - (-1)] dx dy$$

$$\Rightarrow \int_C x dy - y dx = 2 \iint_R dx dy$$

$$\Rightarrow \frac{1}{2} \int_C x dy - y dx = \iint_R dx dy$$

$$\Rightarrow \left[\iint_R dx dy = \frac{1}{2} \oint_C x dy - y dx \right]$$

$$\iint dx dy = \frac{1}{2} \oint x dy - y dx$$

Circle $\iint dx\,dy = \pi r^2$

Ellipse $\iint dx\,dy = \pi ab$

① Verify Green's theorem in a plane $\int_C (xy+y^2)dx + x^2 dy$ ~~$+x^2dy$~~ when 'c' is the closed curve of the region bounded $y=x^2$, $y=x$

Sol:- $\int_C P dx + Q dy = \iint_R (Q_x - P_y)\, dx\, dy \rightarrow ①$

Given $\int_C (xy+y^2)dx + x^2 dy \rightarrow ②$

Let $P = xy + y^2 \quad Q = x^2$

To find double integral,

$P_y = \frac{dP}{dy} = (x+2y)$, $\frac{dQ}{dx} = Q_x = 2x$

$\iint (Q_x - P_y)\, dx\, dy$

$R: y = x^2$
$\quad y = x$

$= \iint_{R: y=x^2,\ y=x} [2x - (x+2y)]\, dx\, dy$

$= \iint_R (x-2y)\, dx\, dy$

if $x=0$, $y=0$

if $x=1$, $y=1$

$x=0$, $x=1$

$y=x^2$, $y=x$

$$\Rightarrow \iint_R (Q_x - P_y)\,dx\,dy = \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=x} (x-2y)\,dy \right] dx$$

$$= \cancel{\int_0^1 \left[\left(\frac{x^2}{2}\right)_{x^2}^{x} - 2y \right.} [x$$

$$= \int_0^1 \left[x\,[y]_{x^2}^{x} - \left[\frac{2y^2}{2}\right]_{x^2}^{x} \right] dx$$

$$= \int_0^1 \left[x\,[x - x^2] - [x^2 - x^4] \right] dx$$

$$= \int_0^1 \left[x^2 - x^3 - x^2 + x^4 \right] dx$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$\int P\,dx + Q\,dy = \int_{\substack{C:\ y=x^2 \\ y=x}} (xy+y^2)\,dx + x^2\,dy$$

$$= \int_{C_1:\ y=x^2} (xy+y^2)\,dx + x^2\,dy + \int_{C_2:\ y=x} (xy+y^2)\,dx + x^2\,dy$$

$$\Rightarrow \mathrm{I} + \mathrm{II}$$

$y = x^2$

$\frac{dy}{dx} = 2x \Rightarrow \boxed{dy = 2x\,dx}$ | $\frac{dy}{dx} = 1$

$\boxed{dy = dx}$

$$I = \int_{C_1 : y = x^2} (xy + y^2)dx + x^2 dy \qquad \boxed{x = 0, x = 1}$$

$$= \int_0^1 \left[x(x^2) + (x^4) \right] dx + x^2 dx . 2x$$

$$= \int_0^1 \left[x^3 + x^4 \right] dx + 2x^3 dx$$

$$= \left(\frac{x^4}{4} \right)_0^1 + \left(\frac{x^5}{5} \right)_0^1 + 2\left(\frac{x^4}{4} \right)_0^1$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{\cancel{2}}{\cancel{4}\,2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{2} = \frac{19}{20}$$

$$II = \int_{C_2 : y = x} (xy + y^2) dx + x^2 dy \qquad \boxed{x = 1, x = 0}$$

$$= \int_1^0 (x(x) + x^2) dx + x^2 dx$$

$$= \int_1^0 (x^2 + x^2 + x^2) dx$$

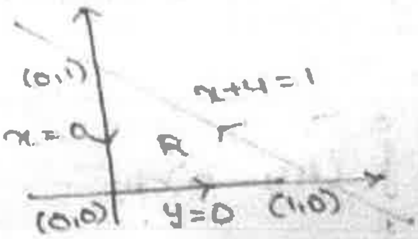
$$= 3\left[\frac{x^3}{3} \right]_1^0 = 3\left(\frac{-1}{3} \right) = -1$$

$$I + II = \frac{19}{20} - \frac{1}{1} \Rightarrow \frac{19-20}{20} = \frac{-1}{20} //$$

L.H.S = R.H.S

$\Rightarrow$ Green's Theorem Verified //

(**) $\int_C (3x^2-8y^2)dx + (4y-6xy)dy \quad y=0 \quad x=0 \quad x+y=1$



Sol:- $\int (3x^2-8y^2)dx + (4y-6xy)dy$

$C: \begin{cases} y=0 \\ x=0 \\ x+y=1 \end{cases}$

$$= \int_{C_1: y=0} (3x^2-8y^2)dx + (4y-6xy)dy + \int_{C_2: x=0} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$+ \int_{C_3: x+y=1} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$I = \int_{x=0}^{x=1} (3x^2-8y^2)dx + (4y-6xy)dy \qquad C_1: y=0$$

$$\frac{dy}{dx} = 0 \qquad \boxed{dy=0}$$

$$= \int_0^1 (3x^2-8y^2)dx + 0 \qquad C_1: y=0$$

$$= \int_0^1 (3x^2-8(0))dx = \int_0^1 3x^2$$

$$= 3\left(\frac{x^3}{3}\right) = 3\left(\frac{1}{3}\right) = 1$$

$$C_2 = \int_1^0 (3x^2-8y^2)dx + (4y-6xy)dy$$

$C_2: x=0$

$\boxed{dx=0}$

$$= \int_1^0 0 + (4y - 6xy)\,dy$$

$C_2 : x = 0$

$$= \int_1^0 (4y - 0)\,dy = \int_1^0 4y\,dy$$

$C_2 : x = 0$ $\qquad$ $C_2 : x = 0$

$$= 0 - \left[\frac{4(1)^2}{2}\right] = -2$$

$$C_3 = \int_1^0 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$C_3 : x + y = 1$

$x + y = 1$

$\boxed{y = 1 - x}$

$$\frac{dy}{dx} = -1 \Rightarrow \boxed{dy = -dx}$$

$$= \int_1^0 (3x^2 - 8(1-x)^2)\,dx + (4(1-x) - 6(x)(1-x))\,(-dx)$$

$$= \int_1^0 \left[3x^2 - 8(1 + x^2 - 2x)\right]dx + \left[4 - 4x - 6x + 6x^2\right](-dx)$$

$$= \int_1^0 \left[3x^2 - 8 - 8x^2 + 16x\right]dx + \left[-4 + 4x + 6x - 6x^2\right]dx$$

$$= \int_1^0 \left[-5x^2 - 8 + 16x\right]dx + \left[-6x^2 + 10x - 4\right]dx$$

$$= \int_1^0 \left[-11x^2 + 26x - 12\right]dx$$

$$= \int_1^0 \left[-11x^2 + 26x - 12\right]dx$$

$$= \left[-11\left(\frac{x^3}{3}\right)_1^0 + 26\left(\frac{x^2}{2}\right)_1^0 - 12(x)_1^0\right]$$

$$= 0 - \left[-\frac{11}{3} + \frac{26}{2} - 12\right]$$

$$= \frac{11}{3} - 13 + 12 = \frac{11}{3} - 1 = \frac{8}{3}$$

$$\therefore C_1 + C_2 + C_3 = 1 - 2 + \frac{8}{3}$$

$$= \frac{8}{3} - 1 = \frac{5}{3}$$

Green's Theorem Verified.

(*) Verify Green's Theorem in Plane

$\int_C (x-2y)dx + xdy$ where c is a Circle $x^2+y^2=1$

Sol:-

$$\int_C Pdx + Qdy = \iint_{\substack{R \text{ on } xy \\ \text{Plane}}} (Q_x - P_y)dxdy$$

$P = x - 2y$	$Q = x$
$\frac{dp}{dy} = -2$	$\frac{dQ}{dx} = 1$

To find $\iint (Q_x - P_y)\,dx\,dy$.

$$\iint_{\substack{R \text{ on } xy \\ \text{plane} \\ (\text{circle})}} (Q_x - P_y)\,dx\,dy = \iint_{R: x^2+y^2=1} (1-(-2))\,dx\,dy = \iint_{R} 3\,dx\,dy$$

$$= 3\iint dx\,dy$$

$$= 3\pi r^2 \text{ if } \boxed{r=1}$$

$$\iint_{\substack{R \text{ on } xy \\ \text{plane} \\ (\text{circle})}} (Q_x - P_y)\,dx\,dy = 3\pi$$

(Another Method on Next Page)

Gauss Divergence Theorem

$$\boxed{\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$

Let ~~'s'~~ 'v' be the volume enclosed by the closed surface 's'.

Let '$\vec{F}$' be a vector point function is continuous with the first order partial derivative exists inside and on the surface 'S'.

The surface integral of $\vec{F}$ over 's' is equal to the volume of the divergence of $\vec{F}$ throughout ~~of~~ the volume 'V'.

Note:

$$\nabla = \vec{i}\,\frac{d}{dx} + \vec{j}\,\frac{d}{dy} + \vec{k}\,\frac{d}{dz}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\boxed{\nabla\cdot\vec{F} = \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz}} \longrightarrow \text{scalar function}$$

$\hat{n}$ is the unit outward normal vector to the surface.

Relation b/w Line Integral & Green's Theorem

Line integral:-

$$\int_C \vec{F}\cdot d\vec{r}$$

Let $\vec{F} = P\vec{i} + Q\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot d\vec{r} = (P\vec{i} + Q\vec{j})\cdot(dx\vec{i} + dy\vec{j})$

$= P\,dx + Q\,dy$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy}$$

Line integral = Green's Theorem

Green's Theorem

$$\int_C P\,dx + Q\,dy = \iint_{R:\,x^2+y^2=1} (Q_x - P_y)\,dx\,dy$$

To Find Line Integral

$$\int_C P\,dx + Q\,dy = \int_C (x-2y)\,dx + x\,dy$$

Use parametric method

$x = r\cos\theta$, $y = r\sin\theta$

$\boxed{\therefore r = 1}$

$x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta \Rightarrow \boxed{dx = -\sin\theta\,d\theta}$

$y = \sin\theta$

$\frac{dy}{d\theta} = \cos\theta$

$\boxed{dy = \cos\theta\, d\theta}$

$\theta = 0 \;,\; \theta = 2\pi$

$$\int_C P\,dx + Q\,dy = \int_C (x-2y)\,dx + x\,dy$$

$$= \int_0^{2\pi} \left[\cos\theta - 2(\sin\theta)\right] - \sin\theta\, d\theta + \cos\theta \cos\theta\, d\theta$$

$$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + 2\sin^2\theta\right] d\theta + \cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \left[-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta\right] d\theta$$

$$\boxed{\begin{array}{l} \sin 2\theta = 2\sin\theta\cos\theta \\ \sin\theta\cos\theta = \frac{\sin 2\theta}{2} \end{array}}$$

$$= \int_0^{2\pi} \left[-\frac{\sin 2\theta}{2} + \sin^2\theta + 1\right] d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} \sin 2\theta\, d\theta + \int_0^{2\pi} \sin^2\theta\, d\theta + \int_0^{2\pi} d\theta$$

$$= -\frac{1}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} + \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta + 2\pi$$

$$= +\frac{1}{4}\left[\cos 2\theta\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} 1\, d\theta - \int_0^{2\pi} \cos 2\theta\, d\theta + 2\pi$$

$$= \frac{1}{4}(1-1) + \frac{1}{2}(2\pi) - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right] + 2\pi$$

$$= 0 + \pi + 0 + 2\pi$$

$$= 3\pi$$

Green's Theorem Verified

Gauss Divergence Theorem

① If 'S' is any closed surface bounded by the volume 'V' and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$. Prove that

$$\iint_S \vec{F}\cdot\vec{n}\,ds = (a+b+c)V$$

Sol:-

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(ax\vec{i} + by\vec{j} + cz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$\nabla\cdot\vec{F} = a+b+c$$

By G.D Theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \iiint_V (a+b+c)\,dv$$

$$= (a+b+c)\iiint_V dv$$

$$= (a+b+c)V$$

$$\therefore \iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv = (a+b+c)V$$

(*) $\vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$ (Cube)

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

By Apply divergence theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

To find volume of Cube

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{d}{dx} + \vec{j}\frac{d}{dy} + \vec{k}\frac{d}{dz}\right)\cdot\left(4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}\right)$$

$$= \frac{d}{dx}(4xz) + \frac{d}{dy}(-y^2) + \frac{d}{dz}(yz)$$

$$\nabla\cdot\vec{F} = 4z - 2y + y = 4z - y$$

Limits

$x=0,\ x=1$

$y=0,\ y=1$

$z=0,\ z=1$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1}\left(\int_{z=0}^{z=1}(4z-y)\,dz\right)dy\right]dx$$

$$= \cancel{\int_0^1\int_0^1 4x[z]_0^1 - y[z]_0^1}$$

$$= \int_0^1\left[\int_0^1 4\left(\frac{z^2}{2}\right)_0^1 - y[z]_0^1\,dy\right]dx$$

$$= \int_0^1\left[\int_0^1\left(\frac{\cancel{4}\,^{2}}{\cancel{2}} - y\right)dy\right]dx$$

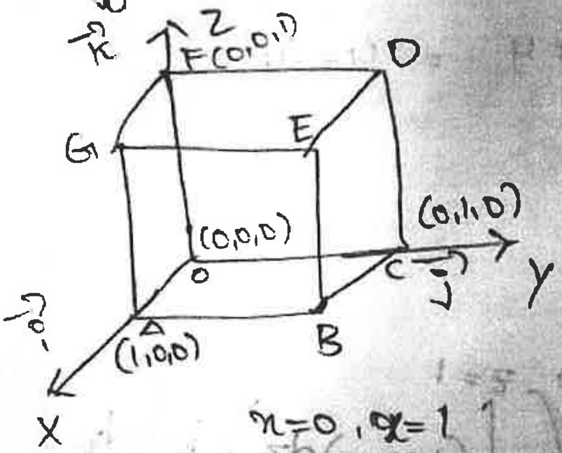
$$= \int_0^1 \int_0^1 (2-y)\,dy\,dx$$

$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx$$

$$= \int_0^1 \left(2(1) - \frac{1}{2}\right) dx = \frac{3}{2}\int_0^1 dx$$

$$= \frac{3}{2} \quad //$$

To find $\iint \vec{F}\cdot\vec{n}\,ds$



$x=0,\ x=1$
$y=0,\ y=1$
$z=0,\ z=1$

Surfaces	$\hat{n}$	$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ $\vec{F}\cdot\vec{n}$	Equation	Value of $\vec{F}\cdot\vec{n}$	ds
S_1: ABEG	$\vec{i}$	$4xz$	$x=1$	$4z$	$dy\,dz$
S_2: OCDF	$-\vec{i}$	$-4xz$	$x=0$	0	$dy\,dz$
S_3: BCDE	$\vec{j}$	$-y^2$	$y=1$	-1	$dx\,dz$
S_4: OAGF	$-\vec{j}$	y^2	$y=0$	0	$dx\,dz$
S_5: GEDF	$\vec{k}$	yz	$z=1$	y	$dx\,dy$
S_6: OABC	$-\vec{k}$	$-yz$	$z=0$	0	$dx\,dy$

$$\therefore \iiint_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_1} \vec{F}\cdot\vec{n}\,ds + \iint_{S_2} \vec{F}\cdot\vec{n}\,ds + \iint_{S_3} \vec{F}\cdot\vec{n}\,ds$$

$$+ \iint_{S_4} \vec{F}\cdot\vec{n}\,ds + \iint_{S_5} \vec{F}\cdot\vec{n}\,ds + \iint_{S_6} \vec{F}\cdot\vec{n}\,ds \longrightarrow (1)$$

$$\Rightarrow \iint_{S_1} \vec{F}\cdot\vec{n}\,ds = \int_{y=0}^{y=1}\int_{z=0}^{z=1} (4z)\,dz\,dy$$

$$= \int_0^1 \left(\frac{4z^2}{2}\right)_0^1 dy$$

$$= 2\int_0^1 dy = 2[y]_0^1 = 2 \,//$$

$$\Rightarrow \iint_{S_2} \vec{F}\cdot\vec{n}\,ds = \int_{y=0}^{y=1}\left[\int_{z=0}^{z=1} 0\right] dy\,dz = 0$$

$$\Rightarrow \iint_{S_3} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=1}\left[\int_{z=0}^{z=1} (-1)\right] dz\,dx$$

$$= \int_0^1 (-1)[z]_0^1\,dx$$

$$= -1\int_0^1 dx = -1 \,//$$

$$\Rightarrow \iint_{S_4} \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1 (0)\,dx\,dz = 0 \,//$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=1}\int_{y=0}^{y=1} y\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

$$\iint_{S_6} \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1 (0)\,dy\,dx = 0$$

From eqn ①

$$\iint_S \vec{F}\cdot\vec{n}\,ds = 2+0-1+0+\frac{1}{2}+0$$

$$= 1+\frac{1}{2} = \frac{3}{2}$$

∴ Gauss Divergence Theorem is Verified.

(*) $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$

$x=0, x=a$ & $y=0, y=b$ & $z=0, z=c$

Sol:- By Gauss Divergence Theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left((x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy)$$

$= 2x + 2y + 2z$

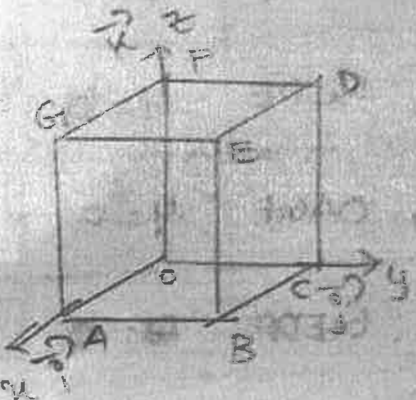
$= 2(x+y+z)$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_{x=0}^{x=a}\left[\int_{y=0}^{y=b}\left(\int_{z=0}^{z=c} 2(x+y+z)\right)dz\,dy\right]dx$$

$$= 2\int_0^a\int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dy\,dx$$

$$= 2\int_0^a\int_0^b \left(cx + cy + \frac{c^2}{2}\right)dy\,dx$$

$$= 2c\int_0^a \left(xy + \frac{y^2}{2} + \frac{cy}{2}\right)_0^b dx$$



$$= 2c\int_0^a \left(bx + \frac{b^2}{2} + \frac{bc}{2}\right)dx$$

$$= 2bc\int_0^a \left(x + \frac{b}{2} + \frac{c}{2}\right)dx$$

$$= 2bc\left[\frac{x^2}{2} + \frac{bx}{2} + \frac{cx}{2}\right]_0^a$$

$$= 2bc\left[\frac{a^2}{2} + \frac{ab}{2} + \frac{ac}{2}\right]$$

$$= \frac{\cancel{2}bc}{\cancel{2}}(a^2+ab+ac)$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

To Find $\iint_S \vec{F}\cdot\vec{n}\,ds$ (L.H.S)

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_{S_1} \vec{F}\cdot\vec{n}\,ds + \iint_{S_2} \vec{F}\cdot\vec{n}\,ds + \iint_{S_3} \vec{F}\cdot\vec{n}\,ds$$

$$+ \iint_{S_4} \vec{F}\cdot\vec{n}\,ds + \iint_{S_5} \vec{F}\cdot\vec{n}\,ds + \iint_{S_6} \vec{F}\cdot\vec{n}\,ds$$

Surfaces	Equation	Unit outward Normal Vector	$\vec{F}\cdot\vec{n}$ $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$	Value of $\vec{F}\cdot\vec{n}$	ds
S_1: ABEG	$x=a$	$n=\vec{i}$	x^2-yz	a^2-yz	$dydz$
S_2: OCDF	$x=0$	$n=-\vec{i}$	$-x^2+yz$	yz	$dydz$
S_3: BCDE	$y=b$	$n=\vec{j}$	y^2-zx	b^2-zx	$dxdz$
S_4: OAGF	$y=0$	$n=-\vec{j}$	$-y^2+zx$	zx	$dxdz$
S_5: GEDF	$z=c$	$n=\vec{k}$	z^2-xy	c^2-xy	$dxdy$
S_6: OABC	$z=0$	$n=-\vec{k}$	$-z^2+xy$	xy	$dxdy$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} (a^2-yz)\,dz\,dy$$

$$= \int_0^b\int_0^c (a^2-yz)\,dz\,dy$$

$$= \int_0^b \left(a^2z - \frac{yz^2}{2}\right)_0^c dy = \int_0^b \left(a^2c - \frac{yc^2}{2}\right) dy$$

$$= \left(a^2cy - \frac{y^2c^2}{4}\right)_0^b$$

$$= a^2cb - \frac{b^2c^2}{4}$$

$$= \frac{4a^2cb - b^2c^2}{4}$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,ds = \int_{y=0}^{y=b}\int_{z=0}^{z=c} (yz)\,dz\,dy$$

$$= \int_0^b \left(\frac{yz^2}{2}\right)_0^c dy = \int_0^b \frac{yc^2}{2}\,dy$$

$$= \left[\frac{y^2}{2}\cdot\frac{c^2}{2}\right]_0^b$$

$$= \frac{b^2c^2}{4}$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,ds = \int_{x=0}^{x=a}\int_{z=0}^{z=c} (b^2 - zx)\,dz\,dx$$

$$= \int_0^a \left(b^2z - \frac{z^2x}{2}\right)_0^c dx$$

$$= \int_0^a \left(b^2c - \frac{c^2x}{2}\right) dx$$

$$= \left(b^2cx - \frac{c^2}{2}\cdot\frac{x^2}{2}\right)_0^a$$

$$= b^2ca - \frac{c^2a^2}{4}$$

Bharath
INSTITUTE OF HIGHER EDUCATION AND RESEARCH
(Declared as Deemed - to - be - University under section 3 of UGC Act 1956)

COURSE FILE

FACULTY	**Dr. R. ANBU**	**FACULTY DEPT**	**MATHS**
SUBJECT	**ADVANCED CALCULUS AND COMPLEX ANALYSIS**	**SUBJECT CODE**	**U20MABT02**
YEAR	**2022 - 2023**	**SEMESTER**	**EVEN**
DEG & BRANCH	**B.TECH**	**DURATION**	**60 Hours**

SL.NO	**DETAILS IN COURSE FILE**	**REMARKS**
1.	LEARNING OUTCOMES	
2.	LESSON PLAN	
3.	CO-PO MAPPING	
4.	INDIVIDUAL TIME TABLE	
5.	SYLLABUS WITH COURSE OUTCOMES	
6.	LECTURE NOTES (FOR ALL UNITS)	
7.	CLA I - QUESTION PAPER	
8.	CLA I - KEY	
9.	CLA I – SAMPLE ANSWER SHEETS	
10.	CLA II - QUESTION PAPER	
11.	CLA II - KEY	
12.	CLA II - SAMPLE ANSWER SHEETS	
13.	CLA III - QUESTION PAPER	
14.	CLA III - KEY	
15.	CLA III - SAMPLE ANSWER SHEETS	
16.	ASSIGNMENT QUESTIONS	
17.	SAMPLE ASSIGNMENTS	
18.	END SEMESTER QUESTION PAPER	
19.	END SEMESTER ANSWER KEY	
20.	TEXT BOOK AND REFERENCE BOOK	
21.	QUESTION BANK	
22.	STUDENT PERFORMANCE RECORD	
23.	STUDENT ATTENDANCE RECORD	
24.	COURSE END SURVEY	
25.	CO ATTAINMENT	

LEARNING OUTCOMES

Subject Name / Code : Advanced Calculus and Complex Analysis / U20MABT02

1. Evaluate multiple integrals using change of variables.
2. Gain knowledge in applying the techniques of vector calculus in problems involving science and engineering in solving ODE.
3. Apply Laplace Transform method for solving many engineering problems.
4. Understand the knowledge in fundamentals of complex analysis functions and its properties.
5. Evaluate improper integrals using Residue Theorem involving problems in science and engineering.
6. Analyze real and complex integrals using the Cauchy's integral formula.

LESSON PLAN

Name of the Department :Mathematics

Name of the School :Schoolof Basic Sciences

Program Name/Code :B.Tech(All branches)/First Year

Academic Year / Semester :2022-2022/ODD

Course Name/Code :Advanced Calculus and Complex Analysis/U20MABT02

a. No. of Credits :4
b. Total Contact Hours :60

Staff Name / ID :Dr. R. Anbu/5166

S.No	Topic	Hr	Reference	Teaching Tool	Proposed Date	Completed Date	BT
UNIT I-MULTIPLE INTEGRALS							
1	Introduction to double integration. Evaluation of double integral when the limits are constant and variable.	1	R1	T6	10-10-22		2
2	Evaluation of double integral (Cartesian forms and Polar forms)	1	R1	T6	11-10-22		2
3	Changing of order of integration in Cartesian coordinates	1	R1	T6	12-10-22		3
4	Tutorial	1	R1	T2	13-10-22		3
5	Area of the region using double integral (Cartesian forms and Polar forms)	1	R1	T6	14-10-22		3
6	Evaluation of triple integral using the Cartesian coordinates and solving the problems	1	R1	T6	17-10-22		2
7	Convert the Cartesian coordinate into Polar coordinate in triple integral, Solving problems in Volume integral	1	R1	T6	18-10-22		2
8	Tutorial	1	R1	T2	19-10-22		2
9	Using triple integral techniques to find the Area	1	R1	T6	20-10-22		3
10	Apply triple integral techniques to find the Volume.	1	R1	T6	21-10-22		3
11	Applications of integral in engineering - Determine the mass an object, Calculate the force exerted of an objects	1	R1	T6	26-10-22		3

12	Tutorial	1	R1	T2	27-10-22		3
UNIT II- VECTOR CALCULUS							
13	Introduction to vector calculus and review of vectors in 2, 3 dimensions. Definition for Gradient-divergence-curl.	1	R2	T6	28-10-22		2
14	Solving problems in Solenoidal& Irrotational.	1	R2	T6	31-10-22		2
15	Directional derivatives. Evaluation of line integrals and surface integrals	1	R2	T6	01-11-22		2
16	Tutorial	1	R2	T2	02-11-22		2
17	Verification of Green's theorem and solve problems using Green's theorem.	1	R2	T6	03-11-22		3
18	State the Gauss divergence theorem and problems for Gauss divergence theoremand explaining how surface integral relates with volume integral	1	R2	T6	04-11-22		3
19	Problems on Gauss divergence theorem for Cube&parallelepiped	1	R2	T6	07-11-22		3
20	Tutorial	1	R2	T2	08-11-22		3
21	State the Stokes's theorem and solving the problems and explaining how line integral relates with double integral	1	R2	T6	09-11-22		3
22	Problems on Stokes's theorem for Cube&parallelepiped	1	R2	T6	10-11-22		2
23	Application of Line and Volume Integrals in Engineering	1	R2	T6	11-11-22		3
24	Tutorial	1	R2	T2	14-11-22		3
UNIT III-LAPLACE TRANSFORM							
25	Introduction to Laplace transforms and Properties of Laplace Transforms	1	R2	T6	15-11-22		2
26	Solving problems using properties. Transforms of derivatives and integrals	1	R2	T6	16-11-22		2

27	Problems on Transforms of derivatives and integrals	1	R2	T6	17-11-22		3
28	Tutorial	1	R2	T2	18-11-22		2
29	Initial value theorem and final value theorem and solving problems	1	R2	T6	21-11-22		2
30	Introduction to Inverse Laplace transforms using partial fractions. Derive Inverse Laplace transforms using shifting theorem	1	R2	T6	22-11-22		2
31	LT and ILT convolution theorem, LT of periodic functions	1	R2	T6	23-11-22		3
32	Tutorial	1	R2	T2	24-11-22		3
33	Applications of LT for solving linear ordinary differential equations of second order with constant coefficient	1	R2	T6	25-11-22		3
34	Solution of integral equation and integral equation involving convolution type	1	R2	T6	28-11-22		3
35	Applications of Laplace transform in engineering- Laplace transform is used to simplify calculations in system modelling	1	R2	T6	29-11-22		3
36	Tutorial	1	R2	T2	30-11-22		3
UNIT IV- ANALYTIC FUNCTION							
37	Introduction of analytic function and its Properties. C-R Equations.	1	R2,R3	T6	01-12-22		2
38	Determination of analytic function using Milne Thomson's method	1	R2,R3	T6	02-12-22		3
39	Problems on Milne Thomson's method	1	R2,R3	T6	05-12-22		3
40	Tutorial	1	R2,R3	T2	06-12-22		3
41	Harmonic function , Explain the Conformal mappings : magnification, rotation, inversion, reflection	1	R2,R3	T6	07-12-22		2

42	Problems on Conformal mappings	1	R2,R3	T6	08-12-22		2
43	Problems on fixed points and cross ratio, Bilinear transformation	1	R2,R3	T6	09-12-22		3
44	Tutorial	1	R2,R3	T2	12-12-22		2
45	Problems on Bilinear transformation	1	R2,R3	T6	13-12-22		3
46	State Cauchy's integral theorem and solving problems	1	R2,R3	T6	14-12-22		3
47	Application of Bilinear transformation and Cauchy's integral in engineering- signal processing and discrete time control theory to transform continuous	1	R2,R3	T6	15-12-22		3
48	Tutorial	1	R2,R3	T2	16-12-22		3
UNIT V-COMPLEX INTEGRATION							
49	Derive the Cauchy's integral formulae and solving the problems	1	R2,R3	T6	19-12-22		2
50	Explain the Taylor's expansions and solving the problems	1	R2,R3	T6	20-12-22		3
51	Explain the Laurent's expansions and solving the problems	1	R2,R3	T6	21-12-22		2
52	Tutorial	1	R2,R3	T2	22-12-22		2
53	Classification of singularities and methods of finding residues	1	R2,R3	T6	23-12-22		2
54	State Cauchy's residue theorem and solving the problems	1	R2,R3	T6	26-12-22		2
55	Explain the Contour integration: Unit circle and solving the problems	1	R2,R3	T6	27-12-22		3
56	Tutorial	1	R2,R3	T2	28-12-22		2
57	Introduction to Semi-circular contour and Explain the problems	1	R2,R3	T6	29-12-22		2

58	Apply Cauchy's residue theorem to find Complex integral along a circular contour C	1	R2,R3	T6	30-12-22		3
59	Semi-circular contour Application of Contour integration in Engineering	1	R2,R3	T6	02-01-23		3
60	Tutorial	1	R2,R3	T2	03-01-23		3

Reference Code	Description
R1	Erwin Kreyszig, "Advanced Engineering Mathematics", 9th Edition, John Wiley & Sons, 2006
R2	B.S.Grewal, "Higher engineering Mathematics", Khanna Publishers, 36th Edition, 2010
R3	Veerarajan.T, "Engineering Mathematics for first years", Tata McGraw-Hill, New Delhi-2008
R4	G.B.Thomas and R.L.Finney, "Calculus and Analytic geometry", 9th Edition, Pearson, 2002
R5	Ramana. B.V, "Higher Engineering Mathematics" , Tata McGraw-Hill, New Delhi, 11th - 2008
R6	N. P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications, 2008

Type Code	Teaching Tool Planned
T1	Power Point Presentation
T2	Tutorial and Problem Solving etc.
T3	Video Presentation
T4	Notes
T5	Models
T6	Black Board
T7	Simulation/Practical etc.

Prepared by	Staff Name Dr.R. Anbu	Signature
Verified by	HOD Dr.G.Mathubala	Signature
Approved by	Dr.M. Sundararajan ProVC (Academics)	Signature

CO-PO MAPPING

List of Pos:

Engineering Graduates will be able to:

PO1 Engineering knowledge Apply the knowledge of mathematics science engineering fundamentals and an engineering specialization to the solution of complex problems.

PO2 Problem Analysis Identify, formula, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural science, and engineering sciences.

PO3 Design/Development of Solution Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the information to provide valid conclusions

PO4 Conduct Investigation of Complex Problems Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5 Modern Tool Usage Create, select, and apply appropriate techniques and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO6 The Engineering and Society Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7 Environment and Sustainability Understand the impact of the professional engineering solution in societal and environmental contexts, and demonstrate the knowledge of and need for sustainable development.

PO8 Ethics Apply ethical principle and commit to professional ethics and responsibility and norms of the engineering practice.

PO9 Individual and Team Work Function effectively as an a member or leader in diverse teams and in multidisciplinary settings.

PO10 Communication Communicate effectively on complex engineering activities with the engineering community and with society at large, such as being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11 Project Management and Finance Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work as a member and leader in a team to manage projects and in multidisciplinary environments.

PO12 Life-Long learning Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

STAFF NAME: Dr. D. Venkatesan, MATHEMATICS

DA \ HR	I 9.00- 9.50	II 9.50- 10.40		III 10.50- 11.40	IV 11.40- 12.30		V 1.30- 2.20	VI 2.20-3.10	VII 3.10-4.00
MON		ACCA A1	**B**			**B**			
TUES			**R**		ACCA A1	**R**			
WED	ACCA A1		**E**			**E**			
THUR			**A**	ACCA A1		**A**			
FRI			**C** **K**	ACCA A1		**C** **K**			

U20MABT02	ADVANCED CALCULUS AND COMPLEX ANALYSIS	L	T	P	C
	Total Contact Periods- 60	3	1	0	4
	Prerequisite :+2				
	Course offering Dept/School: Mathematics & Basic Sciences				
	Total Marks:100				

Co.No	COURSE OUTCOME(Cos)	BT Level
CO1	Gain familiarity in evaluation of multiple integrals using Change of variables	3
CO2	Gain knowledge in applying the techniques of vector calculus in problems involving science and Engineering in solving ODE	3
CO3	Many Engineering problems can be transformed in to problems involving ODE, PDE and Integrals. Laplace transform method and Complex methods can be used for solving them .	3
CO4	Gain knowledge in fundamentals of complex analytic functions and its properties	3
CO5	Gain knowledge in evaluating improper integrals using Residue Theorem involving problems in science and Engineering	3

Mapping/Alignment of Cos with PO& PSO

	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2	PSO3
CO1	3	3										1			
CO2	3	3										2			
CO3	3	3										2			
CO4	3	3	2									1			
CO5	3	3	1									2			

(Tick mark or level of correlation: 3-High, 2-Medium, 1-Low)

Course Content:

UNIT I - MULTIPLE INTEGRALS **(9+3) Hrs**

Evaluation of Double integration Cartesian and plane polar coordinates- Evaluation of double integral by changing of order of integration- Area of double integral (Cartesian) - Area of Double integral (polar) - Triple integration in Cartesian coordinates- Conversation from Cartesian to polar in double integrals - Area of triple integral- Applications of Integral in Engineering.

UNIT II - VECTOR CALCULUS **(9+3) Hrs**

Review of Vectors in 2,3 dimensions- Gradient- divergence - Curl – Solenoidal- Irrotational fields- vector identities (without Proof) -Directional derivatives- Line Integrals- Surface Integrals . Volume Integrals Green's Theorem (without proof) - Gauss divergence theorem (without proof) - Verification Applications to cube- Applications to parallelepiped- Stoke's Theorem (without proof) - Verification - Applications to cubes- Applications to parallelepiped - Application of Line and Volume Integrals in Engineering.

UNIT III - LAPLACE TRANSFORM **(9+3) Hrs**

Laplace Transforms of standard functions- Transforms properties- Transforms of derivatives and integrals- Initial value theorems (without proof) and Final value theorems (without proof) - verification for some problems- Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using shifting theorem - LT convolution theorem- ILT convolution theorem- LT of periodic functions - Applications of LT for solving Linear ordinary differential equations up to second order with constant coefficient – Solution of integral equation and integral equation involving convolution type- Applications of Laplace transform in Engineering.

UNIT IV- ANALYTIC FUNCTIONS **(9+3) Hrs**

Definition of Analytic function- Cauchy Riemann equations- Properties of analytic function - Determination of analytic function using Milne Thomson's Method - Conformal mappings: magnification, rotations, inversion, reflection- Bilinear transformation- Cauchy's Integral theorem(without proof) - Cauchy's integral theorem applications- Application of Bilinear transformation and Cauchy Integral in Engineering

UNIT V - COMPLEX INTEGRATION **(9+3)Hrs**

Cauchy's Integral formulae- Taylor's expansions- Laurent's expansions- Types of poles and Residues - Cauchy's residue theorem (without proof) - Contour Integration: Unit Circle- semicircular contour Application of Contour integration in Engineering

Course Coordinator **HoD/Mathematics**

Dr. R. Anbu

(*) $\int_1^2 \int_3^4 x(x+y)\,dx\,dy$

$= \int_1^2 \int_3^4 (x^2+xy)\,dx\,dy$

$= \int_1^2 \left[\int_3^4 (x^2+xy)\,dy \right] dx$

$= \int_1^2 \left[\int_3^4 x^2\,dy + \int_3^4 xy\,dy \right] dx$

$$= \int_1^2 \left[\int_2^4 x^2(y) + \int_3^4 x\left(\frac{y^2}{2}\right) \right] dx$$

$$= \int_1^2 \left[x^2(1) + x\left(\frac{16}{2} - \frac{9}{2}\right) \right] dx$$

$$= \int_1^2 \left[x^2 + x\left(\frac{7}{2}\right) \right] dx$$

$$= \int_1^2 \left[x^2 dx + x\left(\frac{7}{2}\right) dx \right]$$

$$= \left(\frac{x^3}{3}\right)_1^2 + \frac{7}{2}\left(\frac{x^2}{2}\right)_1^2$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right) + \frac{7}{2}\left(\frac{4}{2} - \frac{1}{2}\right)$$

$$= \frac{7}{3} + \frac{7}{2}\left(\frac{3}{2}\right)$$

$$= \frac{91}{12}$$

(2) $$\int_1^2 \int_x^{x^2} xy \, dy \, dx$$

$$= \int_{x=1}^{x=2} \left[\int_{y=x}^{y=x^2} xy \, dy \right] dx$$

$$= \int_1^2 \left[\int_x^{x^2} x \cdot \frac{y^2}{2} \right] dx$$

$$= \int_1^2 \left[x \int_x^{x^2} \frac{y^2}{2} \right] dx$$

$$= \int_1^2 \left[x \cdot \frac{(x^2 - x)}{2\ \ 2} \right] dx$$

$$= \int_1^2 \left[\frac{x^3}{2} \right] dx$$

$$= \frac{1}{2} \int_1^2 \left[x^3 \right] dx$$

$$= \frac{1}{2} \left(\frac{x^4}{4} \right)_1^2$$

$$= \frac{1}{2} \cdot 4$$

$$= \int_1^2 \left[x \left[\frac{x^2}{2} - \frac{x}{2} \right] \right] dx$$

$$= \int_1^2 \left[\frac{x^5}{2} - \frac{x^3}{2} \right] dx$$

$$= \int_1^2 \frac{x^5}{2} dx - \frac{x^3}{2} dx$$

$$= \frac{1}{2} \int_1^2 \left[\frac{x^6}{6} - \frac{x^4}{4} \right]$$

$$= \frac{1}{2} \left[\frac{64}{6} - \frac{1}{6} \right] - \frac{1}{2} \left[\frac{16}{4} - \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{63}{6} \right] - \frac{1}{2} \left(\frac{15}{4} \right)$$

$$= \frac{63}{12} - \frac{15}{8}$$

$$=$$

(3) $$\int_0^{\pi} \int_0^{x} \frac{\sin x}{x} \, dy \, dx = 2$$

$$= \int_{x=0}^{x=\pi} \left[\int_{y=0}^{y=x} \frac{\sin x}{x} \, dy \right] dx$$

$$= \int_0^{\pi} \left[\int_0^{x} \sin x \cdot \frac{1}{x} \, dy \right] dx$$

$$= \int_0^{\pi} \left[\int_0^{x} \sin x \cdot y \right.$$

(*) $$\int_0^1 \int_1^2 (x^2 + y^2) \, dx \, dy$$

$$= \int_0^1 \left[\int_1^2 (x^2 + y^2) \, dy \right] dx$$

$$= \int_0^1 \left[x^2 \int_1^2 1 \, dy + \int_1^2 y^2 \, dy \right] dx$$

$$= \int_0^1 \left[x^2 (y)_1^2 + \left(\frac{y^3}{3} \right)_1^2 \right] dx$$

$$= \int_0^1 \left[x^2(3-1) + \frac{8}{3} - \frac{1}{3} \right] dx$$

$$= \int_0^1 \left[x^2 + \frac{7}{3} \right] dx$$

$$= \int_0^1 x^2\,dx + \frac{7}{3}\int_0^1 1\,dx$$

$$= \left(\frac{x^3}{3} \right)_0^1 + \left(\frac{7}{3}x \right)_0^1$$

$$= \frac{1}{3} + \frac{7}{3} = \frac{8}{3}$$

(iv) $$\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=1} \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dy \right] dx$$

$$= \cancel{\int_0^1 \left[\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dy \,/\, \int \frac{1}{\sqrt{1-y^2}}\,dy \right] dx}$$

$$= \int_0^1 \left[\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy \right] \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \left[\int_0^1 \sin^{-1} y \right] \left[\sin^{-1} x \right]_1^0$$

$\sin\frac{\pi}{2} = 1$

$\sin^{-1}\left(\sin\frac{\pi}{2}\right) = \sin^{-1}(1)$

$\boxed{\frac{\pi}{2} = \sin^{-1}(1)}$

$$= \int_0^1 \left[\sin^{-1}(1) - \sin^{-1}(0) \right] \left[\frac{dx}{\sqrt{1-x^2}} \right]$$

$$= \int_0^1 \left(\frac{\pi}{2} - 0 \right) \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} \left[\sin^{-1} x \right]_0^1 = \frac{\pi^2}{4}$$

(d) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$

$= \int_1^a \left[\int_1^b \frac{1}{y} dy \right] \frac{1}{x} dx$

$= \int_1^a \left[\log y \right]_1^b \frac{1}{x} dx$

$= \int_1^a \left[\log b - \log 1 \right] \frac{1}{x} dx$

$= \int_1^a (\log b) \frac{dx}{dx}$

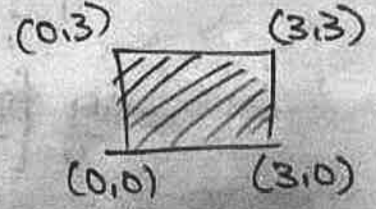
$= \log b \int_1^a \frac{1}{x} dx$

$= \log b \left[\log a - 0 \right]$

$= \log a . \log b .$

(*) $\iint_R x^2 y^2 \, dx\, dy$ over the region bounded by the Straight lines $x=0$, $x=3$ & $y=0$ $y=3$

$= \int_{x=0}^{x=3} \int_{y=0}^{y=3} x^2 y^2 \, dx\, dy$



$= \int_0^3 \left[\int_0^3 y^2 dy \right] x^2 dx$

$= \int_0^3 \left[\frac{y^3}{3} \right]_0^3 x^2 dx$

$= \frac{27}{3} \int_0^3 x^2 dx$

$= \frac{\cancel{27}^{9}}{\cancel{3}} \cdot \frac{\cancel{27}^{9}}{\cancel{3}} = \cancel{9}\ 81$ Sq. units

Area = b×h

= 3×3 = 9

$b = \int_0^3 dx = 3$

$h = \int_0^3 dy = 3$

(ii) Find the area of the double integral

$\iint_R (x-y)\,dy\,dx$ bounded by $y=x$ and $y=x^2$

Sol:-

$y = x \to (1)$

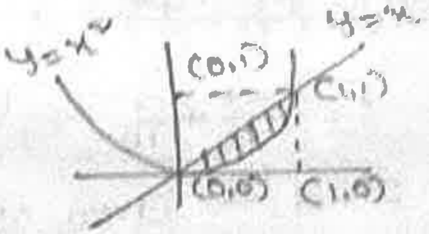
$y = x^2 \to (2)$

$x = x^2$

$x^2 = x \Rightarrow x^2 - x = 0$

$x(x-1) = 0$

$x = 1,\ x = 0$



If $x = 0 \Rightarrow y = 0$

If $x = 1 \Rightarrow y = 1$

Point of intersection (0,0) (1,1)

The unit 'x' varies from $x=0$, $x=1$

The unit 'y' varies from $y=x$, $y=x^2$

$$\iint_R (x-y)\,dy\,dx = \int_{x=0}^{x=1}\left[\int_{y=x^2}^{y=x} (x-y)\,dy\right]dx$$

$$= \int_{x=0}^{x=1}\left[\int_{x^2}^{x} x\,dy - \int_{x^2}^{x} y\,dy\right]dx$$

$$= \int_0^1 \left[x \int_{x^2}^{x} y - \int_{x^2}^{x} \frac{y^2}{2}\, d\right]dx$$

$$= \int_0^1 \left[x(x-x^2) - \left[\frac{x^2}{2} - \frac{x^4}{2}\right]\right]dx$$

$$\cancel{= \int_0^1 \left[x^2 - x^3 - \frac{x^2}{2} + \frac{x^4}{2}\right]dx}$$

$$\cancel{= \int_0^1 \frac{2x^2 - 2x^3 - x^2 + x^4}{2}\,dx}$$

$$\cancel{= \int_0^1 \frac{x^2 + x^4 - 2x^3}{2}\,dx}$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{x^3}{3\times 2} + \frac{x^5}{5\times 2} \right]_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{4}\right) - \frac{1}{6} + \frac{1}{10} \right] - 0$$

$$= \left(\frac{1}{12} - \frac{1}{6}\right) + \frac{1}{10}$$

$$= \frac{-1}{12} + \frac{1}{10} = \frac{12-10}{120} = \frac{1}{60} \text{ Sq.units}$$

(*) Find the area of the double integral $\iint_R dy\,dx$ bounded by $y = x$ and $y = x^2$

Sol:- $y = x \rightarrow (1)$

$y = x^2$

$x = x^2$

$(x - x^2) = 0$

$x(1-x) = 0$

$x = 0,\ x = 1$

If $x = 0,\ x = 1 \Rightarrow y = 0,\ y = 1$

$y = x,\ y = x^2$

The limit 'x' varies from $x = 0,\ x = 1$

The limit 'y' varies from $y = x,\ y = x^2$

$$\iint_R dy\,dx = \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=x} dy \right] dx$$

$$= \int_0^1 \left[y \right]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \left[\frac{1}{2} - \frac{0}{2}\right] - \left[\frac{1}{3} - \frac{0}{3}\right]$$

$= \frac{1}{6}$ Sq. unit.

(*) Find the area of the double integral bounded by the straight line $x=0, y=1, y=x$.

Sol:- $x=0, y=1, y=x$

$x=0 \rightarrow (1)$

$y=1 \rightarrow (2)$

$y=x \rightarrow (3)$

If $y=1 \Rightarrow x=1$ $(1,1)$

The unit varies from $x=0, x=1$

The unit varies from $y=x1, y=x$

$$\iint_R dy\,dx = \int_{x=0}^{x=1}\left[\int_{y=x}^{y=1} dy\right]dx$$

$$= \int_0^1\left[\int_x^1 1\,dy\right]dx$$

$$= \int_0^1 [y]_x^1\,dx$$

$$= \int_0^1 (1-x)\,dx$$

$$= \left(x-\frac{x^2}{2}\right)_0^1$$

$$= 1-0-\left[\frac{1}{2}-\frac{0}{2}\right]$$

$$= 1-\frac{1}{2}$$

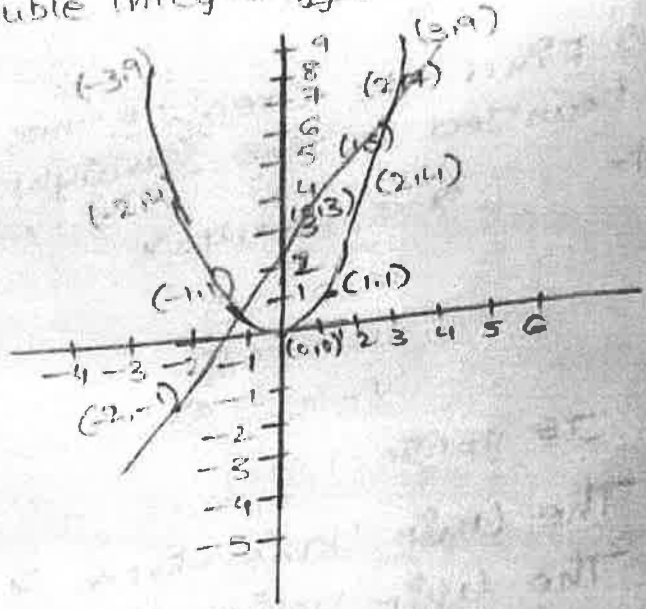
$$= \frac{1}{2} \text{ Sq. units}$$

//

(4) Find Area using Double integral $\iint$ ~~dy dx~~ $y=x^2$ $y=2x+3$

Sol: $y=x^2 \rightarrow (1)$

$y=2x+3 \rightarrow (2)$



$y=x^2$

if $x=0, y=0$ $(0,0)$

$x=1, y=1$ $(1,1)$

$x=-1, y=1$ $(-1,1)$

$x=2, y=4$ $(2,4)$

$x=-2, y=4$ $(-2,4)$

$x=3, y=9$ $(+3,9)$

$x=-3, y=9$ $(-3,9)$

$y=2x+3$

if $x=0, y=3$ $(0,3)$

$x=1, y=5$ $(1,5)$

$x=-1, y=1$ $(-1,1)$

$x=2, y=7$ $(2,7)$

$x=-2, y=-1$ $(-2,-1)$

$y=x^2 \rightarrow (1)$

$y=2x+3 \rightarrow (2)$

$x^2=2x+3$

$x^2-2x-3=0$

$(x+1)(x-3)=0$

$x+1=0 \mid x-3=0$

$\boxed{x=-1}$ $\boxed{x=3}$

if $x=-1 \Rightarrow y=1$ $(-1,1)$

if $x=3 \Rightarrow y=9$ $(3,9)$

The limit x varies from $x=-1, x=3$

The limit y varies from $y=x^2, y=2x+3$

$$\text{Area} = \int_{x=-1}^{x=3}\left[\int_{y=x^2}^{y=2x+3} dy\right]dx = \int_{x=-1}^{x=3}[y]_{x^2}^{2x+3}dx$$

$$= \int_{-1}^{3}(2x+3-x^2)dx$$

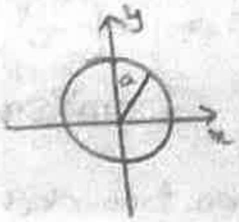
$$= \int_{-1}^{3} 2x\,dx + 3\int_{-1}^{3}dx - \int_{-1}^{3}x^2 dx$$

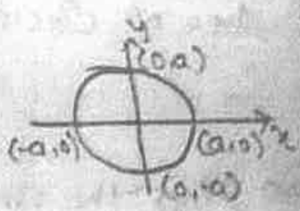
$$= 2\left(\frac{x^2}{2}\right)_{-1}^{3} + 3(x)_{-1}^{3} - \left(\frac{x^3}{3}\right)_{-1}^{3}$$

$$= 2\left(\frac{9}{2}-\frac{1}{2}\right)+3(3-(-1))-\left(9-\left(-\frac{1}{3}\right)\right)$$
$$= 8+3(4)-\left(9+\frac{1}{3}\right)$$
$$= 20-\frac{28}{3}$$
$$= \frac{60-28}{3} = \frac{32}{3} \text{ sq.units}$$

(*) Find the Area of Circle Using Double integral

$x^2+y^2=a^2 \rightarrow (1)$





For x-axis $y=0$

$x^2+0=a^2$

$x^2=a^2 \Rightarrow \boxed{x=\pm a}$

For y-axis $x=$ constant

$x^2+y^2=a^2$

$\boxed{y^2=\pm\sqrt{a^2-x^2}}$ $(0,a)$ $(0,-a)$

if $x=0 \Rightarrow y=\pm\sqrt{a^2}=\pm a$

if $x=\pm a \Rightarrow y=0$

$(a,0)$ $(-a,0)$

The limit x varies from $x=0$, $x=a$

The limit y varies from $y=0$, $y=\sqrt{a^2-x^2}$

$$\left.\begin{array}{l}\text{Area of}\\ \text{1st Quadrant}\end{array}\right\} = \iint dxdy$$

$$= \int_{x=0}^{x=a}\left[\int_{y=0}^{y=\sqrt{a^2-x^2}} dy\right]dx$$

$$= \int_0^a \left[y\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{a^2}{2}\left[\sin^{-1}\left(\frac{a}{a}\right) - \sin^{-1}\left(\frac{0}{a}\right)\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}\right] = \frac{\pi a^2}{4}$$

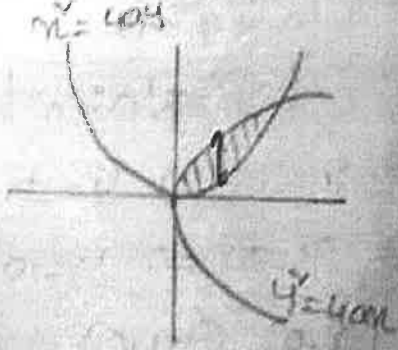
Area of Circle $= 4\left[\frac{\pi a^2}{4}\right]$

$= \pi a^2$ sq.unit

(iv) Find the area bounded by $y^2 = 4ax$ & $x^2 = 4ay$ using double integral.

Sol: $y^2 = 4ax \to ①$, $x^2 = 4ay \to ②$

$\frac{x^2}{4a} = y \to ③$



③ in eqn ①

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\frac{x^4}{16a^2} = 4ax$$

$$x^4 = 64a^3x$$

$$x^4 - 64a^3x = 0$$

$$x(x^3 - (4a)^3) = 0$$

$x = 0$ | $x^3 - (4a)^3 = 0$

$x^3 = (4a)^3$

$x = 4a$

if $x = 0 \Rightarrow y^2 = 4a(0)$

$= 0$

$x = 0 \Rightarrow y = 0$ (0,0)

if $x = 4a \Rightarrow y^2 = 4a(4a)$

$y^2 = (4a)^2$

$y = 4a$

$x = 4a \Rightarrow y = 4a$

$(x, y) = (4a, 4a)$

The limit x varies from $x = 0$, $x = 4a$

The limit y varies from $y = \frac{x^2}{4a}$ to $2\sqrt{ax}$

$$\text{Area} = \iint dx\,dy = \int_{x=0}^{x=4a}\left[\int_{y=\frac{x^2}{4a}}^{y=\sqrt{ax}} dy\right]dx$$

$$= \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}}\,dx$$

$$= \int_0^{4a}\left(2\sqrt{a}\,x - \frac{x^2}{4a}\right)dx$$

$$= \int_0^{4a} 2\sqrt{ax}\,dx - \int_0^{4a}\frac{x^2}{4a}dx$$

$$= 2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left[x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2} - 0\right] - \frac{1}{4a}\left[\frac{(4a)^3}{3} - 0\right]$$

$$= \frac{4\sqrt{a}}{3}\left[4a\sqrt{4a}\right] - \frac{(4a)^3}{3}$$

$$= \frac{16a(2a)}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}\ \text{Sq. units}$$

(*) $\int x^n dx = \dfrac{x^{n+1}}{n+1}$

$\int x^{1/2} dx = \dfrac{x^{1/2+1}}{1/2+1}$

$= \dfrac{x^{3/2}}{3/2} = \dfrac{2}{3} x^{3/2}$ //

Change the order of integration

(1) To evalute $\int_0^\infty \left[\int_x^\infty \dfrac{e^{-y}}{y} dy \right] dx$

Sol:-

Given

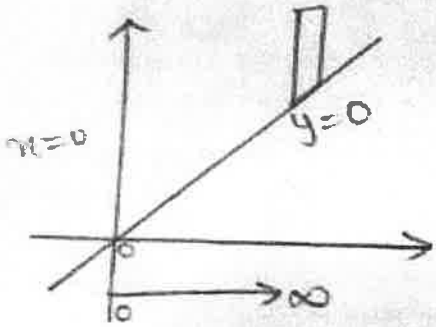
$x : x = 0, x = \infty$

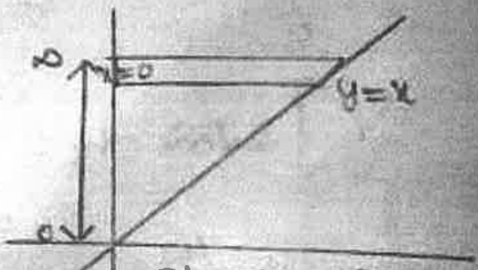
$y : y = x, y = \infty$

Change of order

$y = 0, y = x$

$x = 0, x = y$





change of order

$\Rightarrow \int_{y=0}^{y=\infty} \left[\int_{x=0}^{x=y} \dfrac{e^{-y}}{y} dx \right] dy$

Change of Order

$= \int_0^\infty \dfrac{e^{-y}}{y} \left[\int_0^y dx \right] dy$

$= \int_0^\infty \dfrac{e^{-y}}{y} [x]_0^y dy$

$= \int_0^\infty \dfrac{e^{-y}}{\cancel{y}} (\cancel{y}) dy$

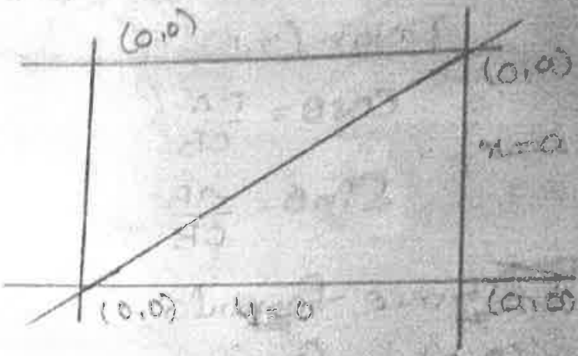
$$= \int_0^\infty e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -1\left[e^{-y}\right]_0^\infty = -1\,(e^{-\infty} - e^{0})$$

$$= -1\,(0-1) = 1$$

(2) $\displaystyle\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$



Given x: $\cancel{x=y}$ $x=y$ to $x=a$

y: $y=0$ to $y=a$

If $y=0 \Rightarrow x=0$ $(0,0)$

If $y=a \Rightarrow x=a$ (a,a)

$x=0$, $x=a$

$y=y$ $\quad y=a$

Change of order

x: $x=0$, $x=a$

y: $y=0$, $y=x$

$$= \int_{y=0}^{y=a}\left[\int_{x=y}^{x=a} \frac{x\,dx}{x^2+y^2}\right]dy$$

$$= \int_{x=0}^{x=a}\left[\int_{y=0}^{y=x} \frac{x}{x^2+y^2}\,dy\right]dx$$

$$= \int_{x=0}^{x=a} x\left[\frac{1}{x}\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)_0^x\right]dx$$

$$= \int_{x=0}^{x=a}\left[\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)_0^x\right]dx$$

$$= \int_{x=0}^{x=a}\left[\mathrm{Tan}^{-1}\left(\frac{y}{x}\right)_0^x\right]dx$$

$$= \int_0^a\left[\mathrm{Tan}^{-1}\left(\frac{x}{x}\right) - \tan^{-1}\left(\frac{0}{x}\right)\right]dx$$

$$= \int_0^a \frac{\pi}{4}\, dx$$

$$= \frac{\pi}{4}[x]_0^a$$

$$= \frac{\pi}{4} \times a \text{ Sq unit}$$

Polar Coordinates

$x^2+y^2 = r^2 \rightarrow (1)$ (Cartesian-form)

Polar (r, θ)

$$\cos\theta = \frac{OA}{OB}$$

$$\sin\theta = \frac{AB}{OB}$$

Distance Formula

$(0,0)$ (x,y)

$$d^2 = (x-0)^2 + (y-0)^2$$

$$d^2 = x^2+y^2$$

$$d = \sqrt{x^2+y^2} = hyp$$

$$r = \sqrt{x^2+y^2} = hyp$$

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

$$x^2+y^2 = r^2$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$$(r^2\cos^2\theta + r^2\sin^2\theta) = r^2$$

$$r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\cos^2\theta + \sin^2\theta = 1$$

$\downarrow$ Circle $\quad \downarrow$ Circle Polar form

Double Integral

Cartesian Coordinate:

- Area $= \iint dx\,dy \rightarrow \left(\int dx\right)\left(\int dy\right)$

$l \times b = $ Area

Polar Coordinate:

- Area $= \iint r\,dr\,d\theta$

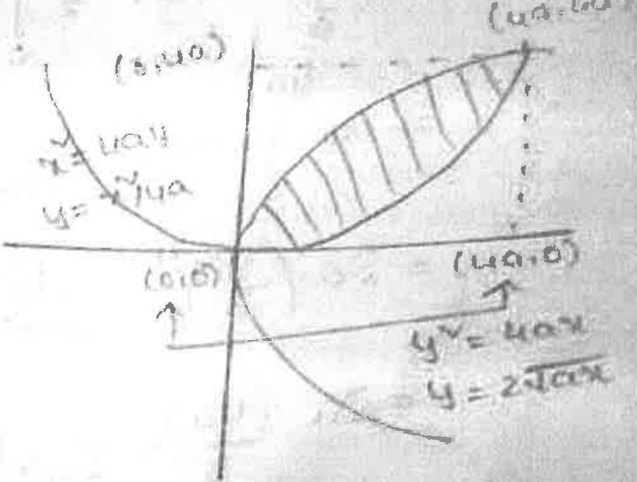
$$= \iint |J|\, dr\, d\theta$$

(3) Change the order of integration

$$\int_0^{4a}\left[\int_{x^2/4a}^{2\sqrt{ax}} dy\right]dx = \int_{x=0}^{x=4a} x\left[\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} y\,dy\right]dx \rightarrow (1)$$

Sol:-

Limits



x : $x=0$, $x=4a$

y : $y=\frac{x^2}{4a}$, $y=2\sqrt{ax}$

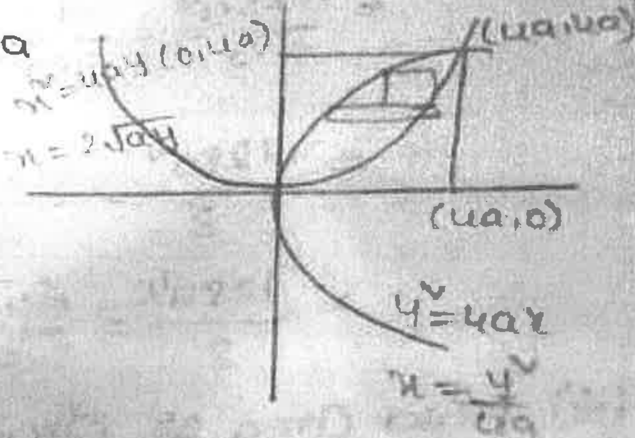
$y=\frac{x^2}{4a} \Rightarrow \boxed{x^2=4ay}$

$y=2\sqrt{ax} \Rightarrow \boxed{y^2=4ax}$

If $x=0 \Rightarrow y=\frac{x^2}{4a}=0$ (0,0)

If $x=4a \Rightarrow y=\frac{(4a)^2}{4a}=4a$

(0,0) (4a,4a)



Change of Order

y : $y=0$, $y=4a$

x : $x=\frac{y^2}{4a}$, $x=2\sqrt{ay}$

$$(1) \Rightarrow \int_{x=0}^{x=4a}\left[\int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} y\,dy\right]dx$$

$$=\int_{y=0}^{y=4a} y\left[\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} x\,dx\right]dy$$

$$=\frac{1}{2}\int_{y=0}^{y=4a} y\left[\left[2\sqrt{ay}\right]^2-\left(y^2/4a\right)^2\right]dy$$

$$= \frac{1}{2}\int_0^{4a}\left[4ay^2 - \frac{y^5}{16a^2}\right]dy$$

$$= \frac{1}{2}\int_0^{4a} 4ay^2 dy - \frac{1}{2}\int_0^{4a}\frac{y^5}{16a^2}dy$$

$$= 2a\int_0^{4a} y^2 dy - \frac{1}{32a^2}\int_0^{4a} y^5 dy$$

$$= 2a\left(\frac{y^3}{3}\right)_0^{4a} - \frac{1}{32a^2}\left(\frac{y^6}{6}\right)_0^{4a}$$

$$= 2a\frac{(4a)^3}{3} - \frac{1}{32a^2}\left[\frac{(4a)^2(4a)^4}{6}\right]$$

$$= \frac{128a^4}{3} - \frac{1}{32a^2}\times 16a^2\frac{(256a^4)}{6}$$

$$= \frac{128a^4}{3} - \frac{128a^4}{6}$$

$$= \frac{128a^4}{3} - \frac{64a^4}{3} = \frac{64}{3}a^4 \text{ Sq.unit.}$$

(4) Find Area of Circle Using Polar Coordinate

Sol:

$$x^2+y^2=a^2$$

$$\sin^2\theta+\cos^2\theta=1$$

Let

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\text{Area} = \iint r\,dr\,d\theta$$

$$\theta=0,\ \theta=2\pi$$

$$r=0,\ r=a$$

$$\text{Area} = \int_0^{2\pi}\left[\int_0^a r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^2}{2}\right)_0^a d\theta$$

$$= \int_0^{2\pi}\frac{a^2}{2}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi}d\theta$$

$$= \frac{a^2}{2}(\theta)_0^{2\pi}$$

$$= \frac{a^2}{\cancel{2}}(\cancel{2}\pi) = \pi a^2 \text{ Sq. unit}$$

(5) $\displaystyle\int_0^a\int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx = \int_{x=0}^{x=a} x\left[\int_{y=\frac{x^2}{a}}^{y=2a-x} y\,dy\right]dx \longrightarrow (1)$

Sol:- Given

$y = \dfrac{x^2}{a} \rightarrow (2)$

$y = 2a - x \rightarrow (3)$

Limits:

$x:\ x=0,\ x=a$

$y:\ y=\dfrac{x^2}{a},\ y=2a-x$

$\left.\begin{array}{l}\text{If } x=0 \text{ then } y=0\ (0,0)\\ \text{If } x=a \text{ then } y=\frac{a^2}{a}=a\ (a,a)\end{array}\right\}\ y=\dfrac{x^2}{a}$

$\left.\begin{array}{l}\text{If } x=0 \text{ then } y=2a\ (0,2a)\\ \text{If } x=a \text{ then } y=a\ (a,a)\end{array}\right\}\ y=2a-x$

$(2) \Rightarrow y = \dfrac{x^2}{a} \Rightarrow x^2 = ay$ (Parabola)

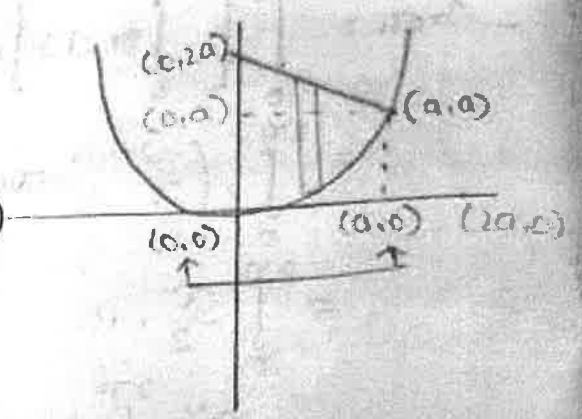
$(3) \Rightarrow y = 2a - x \Rightarrow x+y = 2a$ (St. line)

Apply Change of Order

Limits:

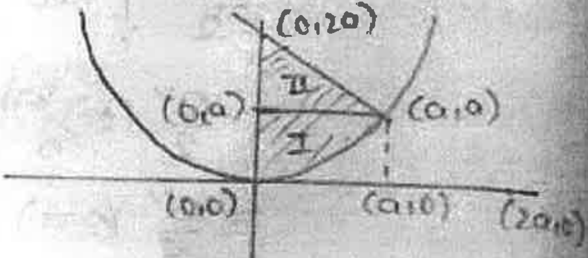
I : $y=0,\ y=a$

$x=0,\ x=\sqrt{ay}$ $\quad \left(\begin{matrix} x^2=ay \\ x=\sqrt{ay}\end{matrix}\right)$



II : $y=a,\ y=2a$

$x=0,\ x=2a-y$



$$① \Rightarrow \int_{x=0}^{x=a} x\left[\int_{y=\frac{x^2}{a}}^{y=2a-x} y\,dy\right]dx$$

$$= I + II$$

$$= \int_{y=0}^{y=a} y\left[\int_{x=0}^{x=\sqrt{ay}} x\,dx\right]dy + \int_{y=a}^{y=2a} y\left[\int_{x=0}^{x=2a-y} x\,dx\right]dy$$

$$I = \int_{y=0}^{y=a} y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy$$

$$= \int_0^a y\left(\frac{ay}{2}\right)dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a$$

$$I = \frac{a}{2}\cdot\frac{a^3}{3} = \frac{a^4}{6}$$

Now $$II = \int_{y=a}^{y=2a} y\left[\int_{x=0}^{x=2a-y} x\,dx\right]dy$$

$$= \int_a^{2a} y\left[\left(\frac{x^2}{2}\right)_0^{2a-y}\right]dy$$

$$= \int_a^{2a} y\left[\frac{(2a-y)^2}{2}\right]dy$$

$$= \int_a^{2a} y\left[\frac{4a^2+y^2-4ay}{2}\right]dy$$

$$= \frac{1}{2}\int_a^{2a}\left[4a^2y+y^3-4ay^2\right]dy$$

$$= \frac{\cancel{4}^{2}a^2}{\cancel{2}}\int_a^{2a} y\,dy + \frac{1}{2}\int_a^{2a} y^3 dy - \frac{\cancel{4}^{2}a}{\cancel{2}}\int y^2 dy$$

$$= 2a^2\left(\frac{y^2}{2}\right)_a^{2a} + \frac{1}{2}\left[\frac{y^4}{4}\right]_a^{2a} - 2a\left[\frac{y^3}{3}\right]_a^{2a} dy$$

$$= 2a^2\left[\frac{(4a^2)}{2}-\frac{a^2}{2}\right] + \frac{1}{2}\left[\frac{16a^4}{4}-\frac{a^4}{4}\right] - 2a\left[\frac{8a^3}{3}-\frac{a^3}{3}\right]$$

$$= \cancel{2}a^2\left[\frac{3a^2}{\cancel{2}}\right] + \frac{1}{2}\left[\frac{15a^4}{4}\right] - 2a\left[\frac{7a^3}{3}\right]$$

$$= 3a^4 + \frac{15a^4}{8} - \frac{14a^4}{3}$$

$$= a^4\left(3+\frac{15}{8}-\frac{14}{3}\right)$$

$$= a^4\left(\frac{72+45-112}{24}\right) = \frac{5a^4}{24}$$

Total Area I + II

$$I + II = \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \cancel{\frac{4a^4}{}} \quad \frac{\cancel{54}^{3}a^4}{\cancel{144}\,8}$$

$$= \frac{3a^4}{8} \text{ Sq. units.}$$

(*) $\displaystyle\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$

$$= \int_{\theta=0}^{\theta=\pi}\left[\int_{r=0}^{r=a} r\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\theta=\pi}\left[\frac{r^2}{2}\right]_0^{a}d\theta = \int_0^{\pi}\frac{a^2}{2}\,d\theta$$

$$= \frac{a^2}{2}\left[\theta\right]_0^{\pi}$$

$$= \frac{a^2}{2}\cdot\pi = \frac{\pi a^2}{2}$$

(*) $\displaystyle\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$

$$= \int_0^{\pi}\left[\int_{r=0}^{r=\sin\theta} r\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta}d\theta = \int_0^{\pi}\frac{\sin^2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\frac{(1-\cos 2\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi}d\theta - \frac{1}{4}\int_0^{\pi}\cos 2\theta\,d\theta$$

$$= \frac{1}{4}(\pi) - \frac{1}{4}\left[\frac{(\sin 2\theta)}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{4} - \frac{1}{4}\cdot\frac{\sin 2\pi}{2}$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4} \text{ Sq. unit.}$$

(*) $\int_0^{\pi/2}\int_0^{\cos\theta} r\,dr\,d\theta$

(*) $\int_0^{\pi}\int_0^{\sqrt{2}} r\,dr\,d\theta$

Sol:- $= \int_0^{\pi}\left[\int_0^{\sqrt{2}} r\,dr\right]d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sqrt{2}} d\theta$

$= \int_0^{\pi}\left(\frac{2}{2}\right) d\theta$

$= [\theta]_0^{\pi} = \pi$

(*) $\int_0^{2\pi}\int_0^{\cos 2\theta} r\,dr\,d\theta$

Sol:- $\int_0^{2\pi}\left[\int_0^{\cos 2\theta} r\,dr\right]d\theta$

$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta$

$= \int_0^{2\pi} \frac{(\cos 2\theta)^2}{2} d\theta$

$= \frac{1}{2}\int_0^{2\pi} (\cos 2\theta)^2 d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{4}(0) = 0$

Cardioid :-

$r = a(1-\cos\theta)$

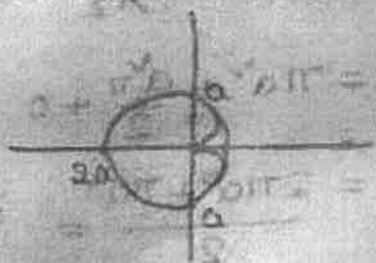
$\theta = 0^\circ \Rightarrow r = a(1-\cos 0^\circ) = 0$

$\theta = \frac{\pi}{2} \Rightarrow r = a(1-\cos\frac{\pi}{2}) = a$

$\theta = \pi \Rightarrow r = a(1-\cos\pi) = 2a$

$\theta = \frac{3\pi}{2} \Rightarrow r = a(1-\cos\frac{3\pi}{2}) = a$

$\theta = 2\pi \Rightarrow r = a(1-\cos 2\pi) = 0$



(*) Area $= \iint r\,dr\,d\theta$

$\theta = 0$, $\theta = 2\pi$

$r = 0$, $r = a(1-\cos\theta)$

$= \int_{\theta=0}^{\theta=2\pi}\left[\int_{r=0}^{r=a(1-\cos\theta)} r\,dr\right]d\theta$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{[a(1-\cos\theta)]^2}{1} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} a^2 (1 + \cos^2\theta - 2\cos\theta)\, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} a^2 + \frac{a^2}{2} \int_0^{2\pi} \cos^2\theta - a^2 \int_0^{2\pi} \cos\theta$$

$$= \frac{a^2}{\cancel{2}} (\cancel{2}\pi) + \frac{a^2}{2} \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} \right) d\theta - a^2 [\sin\theta]_0^{2\pi}$$

$$= \pi a^2 + \frac{a^2}{4} \int_0^{2\pi} d\theta + \int_0^{2\pi} \cos 2\theta \, d\theta - a^2 (0)$$

$$= \pi a^2 + \frac{a^2}{\cancel{4}_2} (\cancel{2}\pi) + \left(\frac{\sin 2\theta}{2} \right)_0^{2\pi}$$

$$= \pi a^2 + \frac{a^2 \pi}{2} + 0$$

$$= \frac{2\pi a^2 + \pi a^2}{2} = \frac{3\pi a^2}{2} \text{ sq. units}$$

(*) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy$ by using polar Coordinate $\to$ ①

Sol÷ Let $x = r\cos\theta$

$y = r\sin\theta$

$$\int_{x=0}^{x=\infty} \int_{y=0}^{y=\infty} e^{-(x^2+y^2)} dy\,dx = \int_{x=0}^{x=\infty} \left(\int_{y=0}^{y=\infty} e^{-(x^2+y^2)} dy \right) dx$$

Limits

x : $x=0$, $x=\infty$

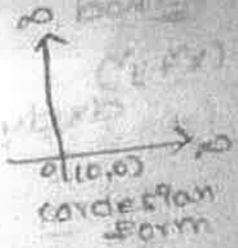
y : $y=0$, $y=\infty$

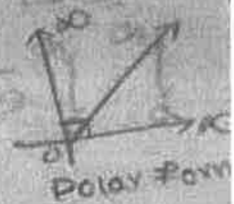
Let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2=r^2 \Rightarrow$ circle with centre $(0,0)$





$\theta=0,\ \theta=\frac{\pi}{2}$

$r=0,\ r=\infty$

① $\Rightarrow \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2}\cdot r\,dr \right) d\theta \rightarrow ②$$

Let $r^2=t \Rightarrow 2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

limits

$r=0,\ t=0$

$r=\infty,\ t=\infty$

② $\Rightarrow \int_0^{\pi/2} \left(\int_0^\infty e^{-t} \frac{dt}{2} \right) d\theta$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\int_0^\infty e^{-t}\,dt \right) d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{e^{-t}}{-1} \right)_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(e^{\infty} - e^{0} \right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1.\,d\theta$$

$$= \frac{1}{2}[\theta]_0^{\pi/2} = \frac{\pi}{4} \text{ Sq. unit}$$

Another Method

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^\infty \int_0^\infty (e^{-x^2} . e^{-y^2}) dx\,dy$$

$$\frac{\pi}{4} = \left(\int_0^\infty e^{-x^2} dx\right) \left(\int_0^\infty e^{-y^2} dy\right)$$

$$\frac{\pi}{4} = \left(\int_0^\infty e^{-x^2} dx\right)^2 \qquad f(x) = y$$

$$\sqrt{\frac{\pi}{4}} = \int_0^\infty e^{-x^2} dx$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-x^2} dx \quad \Bigg| \quad \frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-y^2} dy$$

Triple Integral:-

(1) $$\int_0^a \int_0^b \int_0^c (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^a \int_0^b \int_0^c x\,dx\,dy\,dz + \int_0^a \int_0^b \int_0^c y\,dx\,dy\,dz + \int_0^a \int_0^b \int_0^c z\,dx\,dy\,dz$$

$$= \int_0^a x\left[\int_0^b \left(\int_0^c dz\right) dy\right] dx + \int_0^a \left[\int_0^b y\left(\int_0^c dz\right) dy\right] dx$$

$$+ \int_0^a \left[\int_0^b \left(\int_0^c z\,dz\right) dy\right] dx$$

$$= \int_0^a x\left[\int_0^b (z)_0^c\, dy\right] dx + \int_0^a \left[\int_0^b y\,[z]_0^c\, dy\right] dx$$

$$+\int_0^a\left[\int_0^b\left(\frac{z^2}{2}\right)_0^c dy\right]dx$$

$$= c\int_0^a x\left(\int_0^b dy\right)dx + c\int_{x=0}^{x=a}\left(\int_0^b y\,dy\right)dx + \frac{c^2}{2}\int_0^a\left(\int_0^b dy\right)dx$$

$$= c\int_0^a x\,[y]_0^b\,dx + c\int_0^a \left[\frac{y^2}{2}\right]_0^b dx + \frac{c^2}{2}\int_0^a [y]_0^b\,dx$$

$$= bc\int_0^a x\,dx + \frac{b^2c}{2}\int_0^a dx + \frac{bc^2}{2}\int_0^a dx$$

$$= bc\,\frac{[x^2]^a}{2} + \frac{b^2c}{2}[x]_0^a + \frac{bc^2}{2}[x]_0^a$$

$$= \frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}$$

$$= abc\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}\right) = abc\left(\frac{a+b+c}{2}\right)$$

H.W

(1) $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$ and (2) $\int_0^a\int_0^b\int_0^c (2x+2y+2z)\,dx\,dy\,dz$

(2) Sol:-

$$= \int_0^a x^2\left[\int_0^b\left(\int_0^c dz\right)dy\right]dx + \int_0^a\left[\int_0^b y^2\left(\int_0^c dz\right)dy\right]dx + \int_0^a\left[\int_0^b\left(\int_0^c z^2\,dz\right)dy\right]dx$$

$$= \int_0^a x^2\left[\int_0^b [z]_0^c\,dy\right]dx + \int_0^a\left[\int_0^b y^2[z]_0^c\,dy\right]dx + \int_0^a\left[\int_0^b\left(\frac{z^3}{3}\right)_0^c dy\right]dx$$

$$= c\int_0^a x^2\left[\int_0^b dy\right]dx + c\int_0^a\left[\frac{y^3}{3}\right]_0^b dx + \frac{c^3}{3}\int_0^a\left[\int_0^b dy\right]dx$$

$$= cx^2\int_0^a [y]_0^b dx + \frac{c\cancel{b}^3}{3}\int_0^a dx + \frac{bc^3}{3}\int_0^a dx$$

$$= bc\left[\frac{x^3}{3}\right]_0^a + \frac{b^3c}{3}[x]_0^a + \frac{bc^3}{3}[x]_0^a$$

$$= \frac{a^3bc}{3} + \frac{ab^3c}{3} + \frac{abc^3}{3}$$

$$= abc\left(\frac{a^2+b^2+c^2}{3}\right) //$$

(3) $\int_0^a\int_0^b\int_0^c (2x+2y+2z)\,dx\,dy\,dz$

Sol:-

$$= \int_0^a 2x\left[\int_0^b\left(\int_0^c dz\right)dy\right]dx + \int_0^a\left[\int_0^b 2y\left(\int_0^c dz\right)dy\right]dx + \int_0^a\left[\int_0^b\left(\int_0^c 2z\,dz\right)dy\right]dx$$

$$= 2\int_0^a x\left[\int_0^b \left([z]_0^c dy\right) dx\right] + 2\int_0^a\left[\int_0^b y\,(z)_0^c\,dy\right]dx + 2\int_0^a\left[\int_0^b\left(\frac{z^2}{2}\right)_0^c dy\right]dx$$

$$= 2c\int_0^a x\left(\int_0^b dy\right)dx + 2c\int_0^a\left[\int_0^b y\,dy\right]dx + \frac{\cancel{2}c^2}{\cancel{2}}\int_0^a\left[\int_0^b dy\right]dx$$

$$= 2c\int_0^a x\,[y]_0^b dx + 2c\int_0^a \left[\frac{y^2}{2}\right]_0^b \cancel{dy}\,dx + c^2\int_0^a [y]_0^b dx$$

$$= 2cb\int_0^a x\,dx + \frac{\cancel{2}b^2c}{\cancel{2}}\int_0^a dx + c^2 b\int_0^a dx$$

$$= 2bc\left(\frac{x^2}{2}\right)_0^a + b^2c\,[x]_0^a + bc^2[x]_0^a$$

$$= 2bc\left(\frac{a^2}{2}\right) + b^2c.a + bc^2a$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

$$\boxed{\int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

(4) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

Sol:- $$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}} \right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\,dz \right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}} \right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1)\right] dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx$$

$$= \frac{\pi}{2} \int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy \right] dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2} \left[\frac{x}{1}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right) \right]_0^1$$

$$= \frac{\pi}{2}\left[0 + \frac{1}{2}\sin^{-1}(1)\right] = \frac{\pi}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{8} //$$

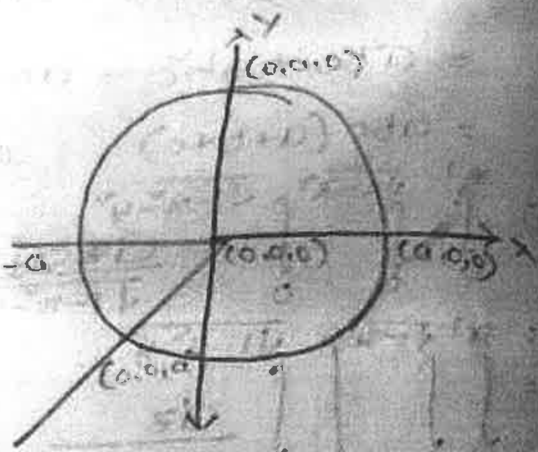
(5) Find the Volume of a Sphere by using triple integrals

Sol:- $x^2+y^2+z^2=a^2 \rightarrow (1)$

for x-axis, $y=z=0$

$x^2=a^2$

$\boxed{x=\pm a}$



For y-axis x-constant

$x=0$ (2D)

$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\pm\sqrt{a^2-x^2}$

if $x=0 \Rightarrow y=\pm\sqrt{a^2-0}$

$=\pm a$

$(0,\pm a)$

if $x=a \Rightarrow y=\pm\sqrt{a^2-x^2}$

$=0$

$(a,0)$

For z-axis

$x=$ constant

$x^2+y^2+z^2=a^2$

$z^2=a^2-x^2-y^2$

$z=\pm\sqrt{a^2-x^2-y^2}$

$$\text{Vol}=\iiint dx\,dy\,dz$$

Limits

x: $x=-a$, $x=a$

y: $y=-\sqrt{a^2-x^2}$, $y=\sqrt{a^2-x^2}$

z: $z=-\sqrt{a^2-x^2-y^2}$, $z=\sqrt{a^2-x^2-y^2}$

$$V=\int_{x=a}^{x=a}\left[\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}}\left(\int_{z=\sqrt{a^2-x^2-y^2}}^{z=\sqrt{a^2-x^2-y^2}} dz\right)dy\right]dx$$

$$\Rightarrow \int_{-1}^{1} x^2 dx = \left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{2}{3} \quad (\text{or}) \quad 2\int_0^1 x^2 dx = \frac{2}{3}$$

$$V=2\int_0^a\left[2\int_0^{\sqrt{a^2-x^2}}\left(2\int_0^{\sqrt{a^2-x^2-y^2}} dz\right)dy\right]dx$$

$$=8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\left[z\right]_0^{\sqrt{a^2-x^2-y^2}} dy\right]dx$$

$$=8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\right]dx$$

$$=8\int_0^a\left[\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\,dy\right]dx$$

$$=8\int_0^a\left[\cancel{\int_0^{\sqrt{a^2-x^2}}}\frac{y\sqrt{a^2-x^2-y^2}}{2}+\frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}}dx$$

(0)

$$=8\int_0^a\left[\frac{a^2-x^2}{2}\sin(1)\right]dx$$

$$=8\int_0^a\left[\frac{\pi}{2}\cdot\frac{a^2-x^2}{2}\right]dx$$

$$=\frac{8\pi}{4}\int_0^a a^2-x^2\,dx \Rightarrow$$

$$=\cancel{\frac{8\pi a^2}{4}\int_0^a x^2 dx}$$

$$=\cancel{2\pi a^2-\left[\frac{x^3}{3}\right]_0^a} \Rightarrow \cancel{2\pi a^2-\frac{a^3}{3}}=\cancel{\frac{5\pi a^2}{3}}a^3$$

$$=2\pi a^2\int_0^a dx-\int_0^a x^2 dx$$

$$=2\pi\left[a^3-\left[\frac{x^3}{3}\right]_0^a\right]$$

$$=2\pi a^3-\frac{a^3}{3}\cdot 2\pi$$

$$=\frac{6\pi a^3-2\pi a^3}{3}=\frac{4\pi a^3}{3}\text{ cubic units}$$

*(6) Volume of ellipse?

Sol:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

For X-axis $y=0=z$

$$\frac{x^2}{a^2}=1$$

$$x^2=a^2 \Rightarrow \boxed{x=\pm a}$$

For Y-axis $x=\text{Const.}$

$z=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right) \Rightarrow \boxed{y=\pm b\sqrt{1-\frac{x^2}{a^2}}}$$

For Z-axis

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\boxed{z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

Limits:

$$x: \; x=-a, \; x=a$$

$$y: \; y=-b\sqrt{1-\frac{x^2}{a^2}}, \; y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$z: \; z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}, \; z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$V=\iiint dx\,dy\,dz$$

$$V=\int_{x=-a}^{x=a}\left[\int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{y=b\sqrt{1-\frac{x^2}{a^2}}}\left(\int_{z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\right)dy\right]dx$$

$$=2\int_{x=0}^{x=a}\left[2\int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}}\left(2\int_{z=0}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\right)dy\right]dx$$

$$=8\int_0^a\left(\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dy\right)dx$$

$$=8\int_0^a\left[\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\cdot\sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}\,dy\right]dx$$

$$=\boxed{\int\sqrt{a^2-x^2}=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$=8c\int_0^a\left[\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{\frac{b^2\cdot\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}{b^2}}\,dy\right]dx$$

$$=\frac{8c}{b}\int_0^a\left[\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}\,dy\right]dx$$

$= \frac{8c}{b}$ let $r = b\sqrt{1-\frac{x^2}{a^2}}$, $r^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$$\boxed{\int \sqrt{a^2-y^2} = \frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)}$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{r^2-y^2}}{2} + \frac{r^2}{2}\sin^{-1}\left(\frac{y}{r}\right)\right]_0^{r=b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y\sqrt{(b^2)\left(1-\frac{x^2}{a^2}\right)-y^2}}{2} + \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}\left(\frac{\cancel{b\sqrt{1-\frac{x^2}{a^2}}}}{b\left(1-\frac{x^2}{a^2}\right)}\right)\right]_0^{y=b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{8c}{2b}\int_0^a \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\sin^{-1}(1)\,dx$$

$$= \frac{\overset{4}{\cancel{8}}cb^{\cancel{2}}}{\cancel{2b}}\int_0^a \left(1-\frac{x^2}{a^2}\right)\frac{\pi}{2}\,dx$$

$$= \frac{\overset{4\,2}{\cancel{8}}\pi cb}{\cancel{2}}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = 2\pi bc\left[[x] - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi bc\left[\frac{3a^3-a^3}{3a^2}\right]$$

$$= 2\pi bc\left[\frac{2a^3}{3a^2}\right]$$

$$= \frac{4\pi abc}{3} \text{ cubic unit.}$$

4) Evaluate $\int_0^1\int_0^2\int_1^2 x^2yz\,dx\,dy\,dz$

$$= \int_{x=0}^{x=1} x^2\left[\int_{y=0}^{y=2} y^2\left(\int_{z=1}^{z=2} z\,dz\right)dy\right]dx$$

$$= \int_0^1 x^2\left[\int_0^2 y\left[\frac{z^2}{2}\right]_1^2 dy\right]dx$$

$$= \int_0^1 x^2 \left[\int_0^2 y \left(\frac{4}{2} - \frac{1}{2} \right) dy \right] dx$$

$$= \int_0^1 x^2 \left[\frac{3}{2} \int_0^2 y \, dy \right] dx$$

$$= \int_0^1 x^2 \left[\frac{3}{2} \left[\frac{y^2}{2} \right]_0^2 \right] dx$$

$$= \int_0^1 x^2 \left[\frac{3}{2} \left(\frac{4}{2} \right) \right] dx$$

$$= \int_0^1 x^2 [3] \, dx = 3 \int_0^1 x^2 dx$$

$$= 3 \left[\frac{x^3}{3} \right] dx = 3 \cdot \frac{1}{3} = 1$$

(8) $\displaystyle\int_0^1 \int_0^2 \int_1^2 x^2 y z \, dx \, dy \, dz$

$$= \int_{z=0}^{z=1} \int_{y=0}^{y=2} \int_{x=1}^{x=2} x^2 y z \, dx \, dy \, dz$$

$$= \int_0^1 z \left[\int_0^2 y \left(\int_1^2 x^2 dx \right) dy \right] dz$$

$$= \int_0^1 z \left[\int_0^2 y \left[\frac{x^3}{3} \right]_1^2 dy \right] dz$$

$$= \int_0^1 z \left[\int_0^2 y \left[\frac{8}{3} - \frac{1}{3} \right] dy \right] dz$$

$$= \frac{7}{3} \int_0^1 z \left[\frac{y^2}{2} \right]_0^2 dz$$

$$= \frac{7}{3} \cdot \frac{4}{2} \int_0^1 z \, dz = \frac{7}{3} \cdot \frac{\cancel{4}^{2}}{\cancel{2}} \left(\frac{1}{\cancel{2}} \right) = \frac{7}{3}$$

(9) $\int_0^1 \int_0^2 \int_1^2 dx\,dy\,dz$

$= \int_0^1 \left[\int_0^2 \left(\int_1^2 dz \right) dy \right] dz$

$= \int_0^1 \left[\int_0^2 [z]_1^2 \, dy \right] dx$

$= \int_0^1 \left[\int_0^2 (1)\, dy \right] dx \;=\; 1 \int_0^1 \left[[y]_0^2 \right] dx$

$= 2\int_0^1 dx = 2$ //

(10) Evoluvate $\int_{-a}^{a} \int_{-b}^{b} \int_{-c}^{c} (x^2+y^2+z^2)\, dx\,dy\,dz$

$= \int_{-a}^{a} x^2 \left[\int_{-b}^{b} \left(\int_{-c}^{c} dz \right) dy \right] dx + \int_{-a}^{a} \left[\int_{-b}^{b} y^2 \left(\int_{-c}^{c} dz \right) dy \right] dx$

$+ \int_{-a}^{a} \left[\int_{-b}^{b} \left(\int_{-c}^{c} z^2 dz \right) dy \right] dx$

$= \int_{-a}^{a} x^2 \left[\int_{-b}^{b} (z)_{-c}^{c} \, dy \right] dx + \int_{-a}^{a} \left[\int_{-b}^{b} y^2 [z]_{-c}^{c} dy \right] dx + \int_{-a}^{a} \left[\int_{-b}^{b} \left(\frac{z^3}{3} \right)_{-c}^{c} dy \right] dx$

$= \int_{-a}^{a} x^2 \left[\int_{-b}^{b} \right.$

(10)
$$\int_{-a}^{a}\left[\int_{-b}^{b}\left(\int_{-c}^{c}(x^2+y^2+z^2)\,dz\right)dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}\left[x^2 z + y^2 z + \frac{z^3}{3}\right]_{-c}^{c} dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}\left(cx^2 + cy^2 + \frac{c^3}{3}\right) - \left(-cx^2 - cy^2 - \frac{c^3}{3}\right)dy\right]dx$$

$$= \int_{-a}^{a}\left[\int_{-b}^{b}\left(2cx^2 + 2cy^2 + \frac{2c^3}{3}\right)dy\right]dx$$

$$= 2c\int_{-a}^{a}\left[\int_{-b}^{b}\left(x^2 + y^2 + \frac{c^2}{3}\right)dy\right]dx$$

$$= 2c\int_{-a}^{a}\left[\left(\cancel{2}yx^2 + \frac{y^3}{3} + \frac{c^2 y}{3}\right)_{-b}^{b} dy\right]dx$$

$$= 2c\int_{-a}^{a}\left[\left[bx^2 + \frac{b^3}{3} + \frac{c^2 b}{3}\right] - \left[-bx^2 + \left(\frac{-b^3}{3}\right) - \frac{bc^2}{3}\right]\cancel{dy}\right]dx$$

$$= 2c\int_{-a}^{a}\left[2bx^2 + \frac{2b^3}{3} + \frac{2bc^2}{3}\right]dx$$

$$= 2c.2b\int_{-a}^{a}\left[x^2 + \frac{b^2}{3} + \frac{c^2}{3}\right]dx$$

$$= 4bc\left[\frac{x^3}{3} + \frac{b^2 x}{3} + \frac{c^2 x}{3}\right]_{-a}^{a}$$

$$= 4bc\left[\left[\frac{a^3}{3} + \frac{b^2 a}{3} + \frac{c^2 a}{3}\right] - \left[\frac{-a^3}{3} - \frac{b^2 a}{3} + \frac{(-c^2 a)}{3}\right]\right]$$

M

Set - B

Name: K. Archana

Section: CSE - N

Roll No: 118 N

Adm No: 8126

Subject code: U20MA BT02

Subject: Advanced calculus and Complex Analysis

Date: 22/11/22

(28) + (2) = (30)

30/30

K. Archana

PART - A

(1) $\int_{1}^{2} \int_{x}^{x^2} xy \, dy \, dx$.

$\int_{1}^{2} x \left(\int_{x}^{x^2} y \, dy \right) dx$.

$\int_{1}^{2} x \left(\int \frac{y^2}{2} \right)_{x}^{x^2} dx$.

$\boxed{\int y \, dy = \frac{y^2}{2}}$

$\int_{1}^{2} x \left(\frac{y^2}{2} \right)_{x}^{x^2} dx$

$\boxed{\int x \, dx = \frac{x^2}{2}}$

$\int_{1}^{2} x \left(\frac{x^4 - x^2}{2} \right) dx$

$\int_{1}^{2} \frac{x^5 - x^3}{2}$

$\left(\frac{1}{2}\right) \left(\frac{x^6}{6} - \frac{x^4}{4} \right)_{1}^{2}$

$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1}}$

$\left(\frac{1}{4}\right) \left(\frac{x^6}{3} - \frac{x^4}{2} \right)_{1}^{2}$

$\left(\frac{1}{4}\right) \left(\frac{2^6}{3} - \frac{2^4}{2} - \left[\frac{1}{3} - \frac{1}{2} \right] \right)$

$\left(\frac{1}{4}\right) \left(\frac{64}{3} - \frac{16}{2} - \frac{1}{3} + \frac{1}{2} \right)$

$\left(\frac{1}{4}\right)\left(\frac{\cancel{63}^{21}}{\cancel{3}} - \frac{15}{2}\right)$

$\left(\frac{1}{4}\right)\left(21 - \frac{15}{2}\right)$

$\left(\frac{1}{4}\right)\left(\frac{42-15}{2}\right)$

$\begin{array}{r} 21 \times 2 = 42 \\ -15 \\ \hline 27 \end{array}$

$\left(\frac{1}{4}\right)\left(\frac{27}{2}\right)$

$\Rightarrow \frac{27}{8}$ //

(2) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$

$\int_{x=1}^{x=a} \int_{y=1}^{y=b} \left(\frac{dy}{y}\right)\left(\frac{dx}{x}\right)$

$\boxed{\int \frac{1}{y} dy = \log y}$

$(\log y)_1^b \int_1^a \frac{dx}{x}$

$(\log b - \log(1))\,(\log x)_1^a$

$(\log b - \log(1))(\log a - \log(1))$

$(\log b)(\log a)$

$\rightarrow (\log a)(\log b)$

(3). Given lines,

$x = 0 \quad , \quad x = 3$

$y = 0 \quad , \quad y = 3$

$\int_{x=0}^{x=3} \int_{y=0}^{y=3} x^2 y^2 \, dx\, dy.$

$\int_{x=0}^{x=3} x^2 \left\{ \int_{y=0}^{y=3} y^2 dy \right\} dx$

$$\int_{x=0}^{x=3} x^2 \left[\int_{y=0}^{y=3} \left(\frac{y^3}{3}\right)\right]_0^3 dx$$

$$\int_{x=0}^{x=3} x^2\left(\frac{27}{3}\right) dx$$

$$\boxed{\int x^n dx \; \frac{x^{n+1}}{n+1}}$$

$$9\int_0^3 x^2 dx$$

$$(9)\left(\frac{x^3}{3}\right]_0^3$$

$$(9)\left(\frac{27}{3}\right)$$

$= 81$ sq units.

(4) find grad ϕ if $\phi = xyz$.

$$\frac{\partial}{\partial x}(\phi) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$\boxed{\frac{\partial}{\partial x}(x) = 1}$$

$$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xyz)$$

$$\boxed{\frac{\partial}{\partial y}(y) = 1}$$

$$(i)\frac{\partial}{\partial x}(xyz) + j\frac{\partial}{\partial y}(xyz) + k\frac{\partial}{\partial z}(xyz)$$

$$\boxed{\frac{\partial}{\partial z}(z) = 1}$$

$\rightarrow (i)(yz) + j(xz) + k(xy)$

at $(1,1,1)$

$\Rightarrow$ $\underline{i + j + k}$

(5) Given surface

$x^2 - y^2 + z = 2$ at point $(1,-1,2)$

$\rightarrow x^2 - y^2 + z - 2 = 0$

Unit vector:

$$\boxed{\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}}$$

$\rightarrow$ whose magnitude is '1'.

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

A:

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 - y^2 + z - 2)$$

$$\Rightarrow i\frac{\partial}{\partial x}(x^2 - y^2 + z - 2) + j\frac{\partial}{\partial y}(x^2 - y^2 + z - 2) + k\frac{\partial}{\partial z}(x^2 - y^2 + z - 2)$$

$$\Rightarrow (i)\{2x\} + j(-2y) + k(1)$$

$$\Rightarrow 2xi - 2yj + k$$

at $(1, -1, 2)$

$$\boxed{\nabla\phi \Rightarrow 2i + 2j + k}$$

$$|\nabla\phi| = \sqrt{4+4+1}$$
$$= \sqrt{9} = 3$$

Unit Vector $= \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2i+2j+k}{3} \,//.$$

Part-B

7(A)

$$\bar{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\nabla\cdot\bar{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2i + y^2j + z^2k)$$

$$\Rightarrow i\frac{\partial}{\partial x}(x^2i + y^2j + z^2k) + j\frac{\partial}{\partial y}(x^2i + y^2j + z^2k)$$
$$+ k\frac{\partial}{\partial z}(x^2i + y^2j + z^2k)$$

$$\Rightarrow \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\boxed{\vec{i}\cdot\vec{i} = 1}$$
$$\boxed{\vec{j}\cdot\vec{j} = 1}$$
$$\boxed{\vec{k}\cdot\vec{k} = 1}$$

$$\Rightarrow 2x + 2y + 2z$$

$$\Rightarrow 2(x+y+z)$$

$$\frac{\partial}{\partial x}(x^2) = 2x$$

$$\frac{\partial}{\partial y}(y^2) = 2y$$

$$\frac{\partial}{\partial z}(z^2) = 2z.$$

$\nabla \times \vec{F} \Rightarrow$

<u>curl $\times \vec{F}$</u>

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

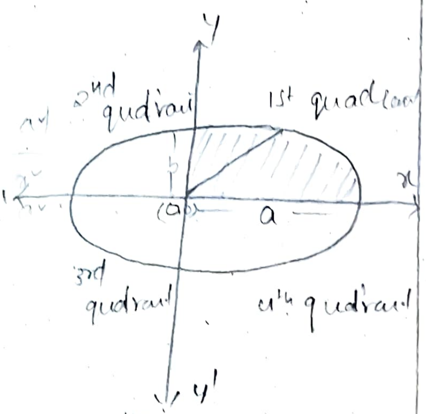
$$\Rightarrow i\left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right) - j\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right) + k\left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right)$$

$$\Rightarrow i(0-0) + j(0-0) + k(0-0)$$

$$\Rightarrow \vec{0}$$

6 (B) Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Major axis a

Minor axis b

for first quadrant we have to find the area.

let $b=0$

then

$$\frac{x^2}{a^2} + 0 = 1$$

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$\boxed{x = \pm a}$$

x- limits varies from $-a$ to $+a$.

Sub 'x' in

<u>y- limits:</u>

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \Rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

y limits varies from
$y = -b\sqrt{1-\frac{x^2}{a^2}}$ to $+b\sqrt{1-\frac{x^2}{a^2}}$

so $\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$ → can also be written as

$$\to 2\int_0^a 2\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$

$$\Rightarrow 4\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx.$$

$\int dy = y.$

$$4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx.$$

$\int dx = x.$

$$\Rightarrow (4)\int_0^a \left(b\sqrt{1-\frac{x^2}{a^2}} - 0\right) dx$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow (4)\int_0^a \left(b\sqrt{1-\frac{x^2}{a^2}}\right) dx$$

$$(4)\int_0^a \left(\frac{b}{a}\right)\left(\sqrt{a^2-x^2}\right) dx$$

$$\Rightarrow \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\Rightarrow \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]_0^a$$

$$= \frac{4b}{a}\left(\frac{a\sqrt{a^2-a^2}}{2} + \frac{a^2}{2}\sin^{-1}(a/a)\right)$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}(\pi/2)\right]$$

$\left(\frac{4b}{a}\right)\left(\frac{a^2\pi}{4}\right)$

$\Rightarrow$ πab sq units.

$\therefore$ Area of the ellipse is πab sq. units.

Part - C

8(b) Given

ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Let $y=0, z=0$

$$\frac{x^2}{a^2}=1$$

$$\boxed{x=\pm a}$$

Let $z=0$

then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\boxed{y=\pm b\sqrt{1-\frac{x^2}{a^2}}}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$\boxed{z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

we can rewrite the integral as

$$2\int_0^a 2\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} 2\int_0^{c\sqrt{1-\frac{x^2}{a^2}-y^2/b^2}} dz\,dy\,dx.$$

$$8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx.$$

$$8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-x^2/a^2-y^2/b^2}}\, dy\,dx.$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$8c\int_0^a\left(\int_0^{b\sqrt{1-x^2/a^2}}\underbrace{\sqrt{1-x^2/a^2-y^2/b^2}}_{a=\left(\sqrt{1-\frac{x^2}{a^2}}\right)}\,dy\right)dx.\qquad x = y/b$$

$$\boxed{\sqrt{a^2-x^2} = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$\Rightarrow 8cb\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\left(\frac{y}{b}\right)\sqrt{\left(\sqrt{1-\frac{x^2}{a^2}}\right)^2-\frac{y^2}{b^2}}+\frac{1-\frac{x^2}{a^2}}{2}\sin^{-1}\left(\frac{y/b}{\sqrt{1-\frac{x^2}{a^2}}}\right)$$

$$\Rightarrow 8cb\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left(\frac{y}{b}\right)\left(\cancel{\sqrt{1-\frac{x^2}{a^2}-1+\frac{x^2}{a^2}}}\right)^{0}+\frac{1-\frac{x^2}{a^2}}{2}\sin^{-1}\overbrace{\left(\frac{\left(\frac{1}{b}\right)b\sqrt{1-\frac{x^2}{a^2}}}{\sqrt{1-\frac{x^2}{a^2}}}\right)}^{\sin^{-1}(1)}$$

$$\Rightarrow 8cb\int_0^a\int_0^b\frac{a^2-x^2}{2a^2}\sin^{-1}(1)$$

$$\cancel{8c\int_0^a\int_0^b\frac{a^2-x^2}{a^2}}\qquad\Rightarrow 8bc\int_0^a\frac{a^2-x^2}{2a^2}\left(\frac{\pi}{2}\right)$$

$$\frac{8bc\pi}{4}\int_0^a\left(\frac{a^2-x^2}{a^2}\right)dx$$

$$\frac{\pi bc}{2}\int_0^a\left(1-\frac{x^2}{a^2}\right)dy$$

$$\frac{\pi}{2}bc\int_0^a dx-\frac{1}{a^2}\int(x^2)\,dx$$

$$\frac{\pi}{2}bc\left([x]_0^a-\frac{1}{a^2}\left[\frac{x^3}{3}\right]_0^a\right)$$

10

A.A.

$$\frac{\pi}{2} bc\left[a - \frac{1}{\cancel{a^2}}\left(\frac{a^{\cancel{3}}}{3}\right)\right] = \frac{\pi}{2} bc\left[a - a/3\right]$$

$$\frac{\pi}{2} bc\left(\frac{2a}{3}\right)$$

$$\frac{\pi abc}{3} \text{ sq. units.}$$

Roll. No: 71N

Ad: No: 6567

Name: A. Hema Krishna

Subject: U20MAB T02 - ADVANCED CALCULUS AND COMPLEX ANALYSIS

Date: 22.11.2022

SET - B

8

08 / 30

09 / 30

A. Hema Krishna

8) a) $$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx.$$

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx.$$

$\iint \cancel{x^2 y^2}\,dx\,dy$. $x=0,\ x=3,\ y=0,\ y=3.$

$\iint x^2y^2\,dx\,dy$

$\iint 0*3+0*3\,dx\,dy$

$\iint 3+3\,dx\,dy$

$\iint 6\,dx\,dy$

over the region bounded by the integral

$6\,dx\,dy$.

5) $x^2 - y^2 + z = 2$

Point $= (1, -1, 2)$

$x^2 - y^2 + z = 2$

$(1)^2 - (-1)^2 + 2 = 2$

$1 + (-1) + 2 = 2$

$+2 + 2 = 2$

$4 = 2$

$\frac{4}{2} 2$

$= 2$ //

1) $\int_1^2 \int_x^{x^2} xy \, d\cancel{y} \, dx.$

$= \frac{1}{2} \int_0^a$

$\int_1^2 \int_x^{x2} xy \, dy \, d\cancel{x}$

double integral

$\int_1^2 xy \, dy \, dx$

$\int_1^2 \int_x^{x2} x y \, dy \, dx.$

8)a) $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx.$

Sol

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \, dy \, dx.$$

$$\int_{x=0}^{x=a} x \left(\int_{y=\frac{x^2}{a}}^{y=2a-x} y \, dy \right) dx$$

$$= \int_{x=0}^{x=a} x \left(\int_{y=\frac{1}{a}x}^{y=2a-x} y \, dy \right) dx$$

x : $x = a$, $x = a$

y : $y = \frac{x}{4}$, $y = 2a + x$

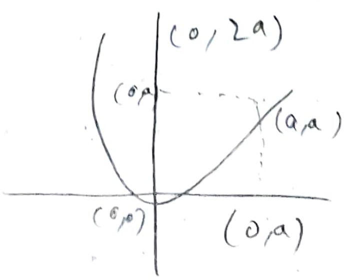
$x = 0$ then $y = a$ $(0,0)$.

$x = 0$ then $y = \frac{x^2}{a} = \frac{0^2}{a} = (0,0) \to (1)$

$x = a$ then $y = \frac{x^2}{a} = \frac{a^2}{a}$ $y = a$ $(a,0) \to (2)$

$x = 0$ then $y = 2a - x$ $(0, 2a) \to (3)$

$x = 0$ then $y = 2a - a$
$= a$ $(a, a) \to (4)$



$$\int_0^a x\left[\int_{\frac{x^2}{a}}^{2a-x} y\,dy\right)dx$$

then
$$\int_{x=0}^{x=a} x\left(\int_{y=\frac{x^2}{a}}^{y=2a-x} y\,dy\right)dx =$$

Change of order

$$= \int_0^a y\int_0^{\sqrt{ay}} x\,dx)\,dy$$

$$= \int_{y=0}^{y=a} y\left(\int_{x=0}^{x=\sqrt{ay}} x\,dx\right)dy$$

$$= \int_{y=0}^{y=a} y\left[\frac{x^2}{2}\right)_0^{\sqrt{ay}} dy$$

$$= \int_{y=0}^{y=a} y\left[\left(\frac{\sqrt{ay}}{2}\right)^2 - 0\right]dy$$

$$= \int_{y=0}^{y=a} y\left\{\frac{ay}{2}\right]dy$$

$$= \frac{a}{2}\int_{y=0}^{y=a} \frac{y^3}{3}$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{a}{2}\left[\frac{a^3}{3}\right) = \frac{a^4}{6} \quad \boxed{I = \frac{a^4}{6}}$$

5/1

$$= \frac{1}{4} \int_{y=0}^{y=2a} \cancel{4}a^2 y^2$$

$$= \frac{1}{4} \int_{y}^{y=2a} 4a^2 y^2) + [y^4]_0^{2a} - 4[ay^3]_0^{2a}$$

$$= \frac{1}{4} \; 4[a^2 y^2]_0^{2a} + [y^4]^{2a} - 4[ay^3]_0^{2a}$$

$$= \frac{1}{4} \quad 4 \; 2a$$

$$\frac{1}{2} \int_{y=0}^{y=2a} y \, (4a^2 + y^2 - 4ay) \, dy$$

$$= \frac{1}{2} \left[\int_0^{2a} 4a^2 y \, dy + \int_0^{2a} y^2 \, dy - \int_0^{2a} 4ay \, dy \right]$$

$$= \frac{1}{2} \left[4^2 \int_0^{2a} a^2 \frac{y^2}{2} + \int_0^{2a} \frac{y^3}{3} - 4 \int_0^{2a} a \frac{y^2}{2} \right]$$

$$= \frac{1}{2} \left[4a^2 \left[\frac{y}{2}^2 \right]_0^{2a} + \left[\frac{y}{3}^3 \right]_0^{2a} - 4a \left[\frac{y^2}{2} \right]_0^{2a} \right.$$

$$= \frac{1}{2} \left[4^2 a^2 \frac{(2a)^2}{\cancel{2}} + \frac{(2a)^3}{3} - 4a \left[\frac{(2a)^2}{2} \right) \right.$$

$$= \frac{1}{2} \left[4a^2 \left(\frac{4a^2}{2} \right) + \frac{8a^3}{3} - \frac{16a^3}{2} \right]$$

$$= \frac{1}{2} \quad \frac{16a^4}{2} + \frac{8a^3}{3} - \frac{16a^3}{2}$$

3) $\int\int x^2y^2\,dx\,dy$

$x=6,\ x=2,\ y=0,\ y=3$

$\int 2x\ 2y$

$\int\int 2(1)2(y) = \int\int 2,2$

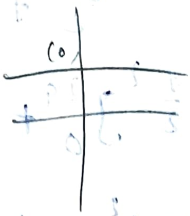
$\int\int 2(3)2(3) - \int\int 6+6$

6) area using double integral

a) $x=0\quad y=1$ and $y=x$

$x\ y\ \cancel{x=1}$

b)



given lines are

$x=0\ (0\ y) \rightarrow ①$

$y=1 \rightarrow ②$

$y=x \rightarrow ③$

$\cancel{y=x}$ —

put ① in ②

$y=0$

$(0,0)$

6).a Put ② in ①

$x = 3$

$(1,1)$

Point of intersection $(1,1)$ & $(0,0)$

$x = 0, \; x = 1$

$y = x, \; y = 1$

$$= \int_{x=0}^{x=1} \left(\int_{y=1}^{y_2} dy \right) dx$$

$$= \int_{x=0}^{x=1} \left(y \right]_x^1 dx$$

$$= \int_{x=0}^{x=1} 1 \, dx - \int_{x=0}^{x=1} x \, dx$$

$$= \left[x \right]_0^1 - \left[\frac{x^2}{2} \right]_0^1$$

$$= \left[1 - 0 \right] + \frac{\left[1 - 0 \right]}{2}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

3

8) b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

for x, axis $y, z = 0$

$\frac{x^2}{a^2} + 0 + 0 = 1$

$x^2 = a^2$

For y-axis, x is const

$z=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+0=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$$

3

for z - x, y are constants

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$\int_{x=a}^{x=a}\int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{y=b}\int_{z=c\sqrt{1-\frac{x^2}{a}-\frac{y^2}{b^2}}}^{z=c} dx\,dy\,dz$$

$$=2\int_{x=0}^{x=a}x\int_{y=0}^{y=b}\int_{z=0}^{z=c} dz\,\Big)dy\Big)dx$$

$$=8\int_{x=0}^{x=a}\left(\int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}}[z]^{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\right.$$

Name:- M. Ganesh

Rollno :- 55-N

Subject code:- U20MABT02

Date :- 15.11.22

CLA - 1

18

$\frac{18}{30}$

M. Ganesh

part-c

8 (B) volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$= \frac{z^2}{c^2} = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

$$\Rightarrow z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right.$$

$$\cancel{z = c}$$

$$z = c\left(\sqrt{1-\frac{x^2}{a^2}}-\frac{y^2}{b^2}\right.$$

xy plane $\rightarrow z=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{y^2}{b^2}=1-x^2/a^2$$

$$y = b\sqrt{1-x^2/a^2}$$

x- plane $y=z=0$

$$\frac{x^2}{a^2}=1 \quad \therefore x^2=a^2 \Rightarrow \boxed{x=a}$$

$$V = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\, dy\,dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\,dx$$

$$= 8c \int_0^a \int_0^{bt} \sqrt{t^2-\frac{y^2}{b^2}}\, dy\,dx$$

$$8c \int_0^a \int_0^{bt} \frac{\sqrt{b^2t^2-y^2}}{b^2}\, dy\,dx$$

$$\frac{8c}{b} \int_0^a \left(\frac{y^2t^2}{2}\left(\sin^{-1}(y/bt)\right) + y/2\sqrt{b^2t^2-y^2}\right) dx$$

$$8bc \times \pi/4 \int_0^a t^2\, dx$$

$$2bc\pi \int_0^a 1-\frac{x^2}{a^2}\, dx$$

$$2b\cancel{b}c\pi \left(x-\frac{x^3}{3a^2}\right]_0^a$$

$$2bc\pi \left(a - \cancel{\quad}\right]_{0-0\,0}^{a}$$

$$\Rightarrow V = \frac{4\pi}{3} abc$$

10

6.

$x=0, \quad y=1, \quad y=x$

given lines

$x=0 \quad (0,) \quad -①$

$y=1 \quad - ②$

$y=3 \quad - ③$

put eq ① in

$y=0, \; (0,0)$

put 2 in eq ③

$x=3$

$(1,1)$

point of iner section

$x=0, \; x=1$

$y=x, \; y=1$

$$= \int_{x=0}^{x=1} \int_{y=1}^{y2} dy \, dx$$

$$= \int_{x=0}^{x=1} 1 \, dx - \int_{x=0}^{x=1} x \, dx$$

$$= \int (x)_0^1 - \left[\frac{1-0}{2}\right] \Rightarrow 1 - \frac{1}{2} = \frac{1}{2}$$

PART-B

7. $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$.

$x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$z^2\vec{k} = x^2\vec{i} + y^2\vec{j}$

$x^2\vec{i} - y^2\vec{j}$

7(B)

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$,

$\nabla r^n = n r^{n-2}\vec{r}$

$= n r^{n-2}\ x\vec{i} + y\vec{j} + z\vec{k}$

$= n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$

$= n r^{n-2}(x+y+z)(\vec{i} + \vec{j} + \vec{k})$

$= n r^{n-2}(xyz)(\vec{i}\,\vec{j}\,\vec{k})$

$= n r^{(n-2)}(x+y+z)(\vec{i}\,\vec{j}\,\vec{k})$

$= (x^2+y^2+z^2)(\vec{i}\,\vec{j}\,\vec{k})$

$= (x\vec{i} + y\vec{j} + z\vec{k})$

$\therefore \nabla r^n = n r^{n-2}\vec{r}$.

Hence proved.

PART-A

4. $\phi = x^2 - y^2 + z = 2$ at $(1,-1,2)$

Unit normal vector $\hat{n} = \frac{\nabla\phi}{|\vec{n}|}$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + z - 2)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 - y^2 + z - 2) + \vec{j}\frac{\partial}{\partial y}(x^2 - y^2 + z - 2) + \vec{k}\frac{\partial}{\partial y}(x^2 - y^2 - z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(1)$$

$$= 2x\vec{i} - 2y\vec{j} + \vec{k}$$

at $(1,-1,2) = 2\vec{i} + 2\vec{j} + \vec{k} =$

$$|\nabla\phi| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\hat{n} = \frac{(2x\vec{i} - 2y\vec{j} + \vec{k})\ (2\vec{i} + 2\vec{j} + \vec{k})}{3}$$

$$= \frac{10}{3}$$

$$\int_1^2 \int_x^{x^2} \cancel{xy\,dy}\,dx$$

1. $$\int_1^2 \int_3^4 x(x+y)\,dx\,dy$$

$$\int_1^2 \int_3^4 (x^2+xy)\,dx\,dy$$

$$= \int_1^2 \left[\int_3^4 (x^2+xy)\,dx\right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_3^4 dy$$

$$= \int_1^2 \left[\frac{4^3}{3} + \frac{4^2 y}{2}\right] - \left[\frac{3^3}{3} + \frac{4^2 y}{2}\right] dy$$

$$= \int_1^2 \left(\frac{64}{3} + \frac{16y}{2} - \frac{27}{3} + \frac{4^2 y}{2}\right)$$

$$= \int_1^2 \frac{64}{3} + 8y - 9 - \frac{8y}{2}\Big)$$

$$= \int_1^2 \frac{64}{3} + 8y - 9 - \frac{8y}{2}\Big)$$

$$= \left\{\frac{64y}{3} - 8y^2 - 9y + \frac{8y^2}{4}\right)$$

Hence Proved.

Register no: u22EC136
Roll No no: 6
Name: G. Dhanashekar
Subject: maths
Section: G-1

Set-B

(2) given

(a) $L[\sinh at] = \frac{a}{s^2-a^2}$

$\boxed{\text{we known that } \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}}$

$= \because e^{\theta}-e^{-\theta} \quad \frac{e^{at}-e^{-at}}{2}$

$= \frac{1}{2}[e^{at}-e^{-at}]$

$= \frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$

$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right)$

$= \frac{a}{s^2-a^2}$ //

(3) (a) given $L\left[a\sqrt{t} + \frac{b}{\sqrt{t}} + c\right]$

$= a\,L[\sqrt{t}] + b\,L[t^{-1/2}] + L[c]$

$a\,L[t^{1/2}] + b\,L[t^{-1/2}] + L[c]$

$= a\,\frac{\sqrt{\pi}}{2s^{3/2}} + b\,\frac{\sqrt{\pi}}{s^{3/2}} + c\,\frac{1}{s}$

$\boxed{\text{we known that } t^{1/2} = \frac{\sqrt{\pi}}{2s^{3/2}}, \quad t^{-1/2} = \frac{\sqrt{\pi}}{s^{3/2}}}$

$= \frac{a\sqrt{\pi}}{2s^{3/2}} + \frac{b\sqrt{\pi}}{s^{3/2}} + \frac{c}{s}$ //

(4) (a) given $L[\sin 5t \cos 3t]$

$\boxed{\sin A \cos B = \frac{\sin(A+B)+\sin(A-B)}{2}}$

$= L[\sin(5t+3t) - \sin(5t-3t)]$

$= L[\sin 8t - \sin 2t]$

$= L\{\sin(8t)\} - L\{\sin 2t\}$

$= \frac{8}{s^2+64} - \frac{2}{s^2+4}$ //

⑤ given

$L^{-1}\left[\frac{3s+2}{s^2-4}\right]$

$= L^{-1}\left[\frac{3s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right]$

$= 3L^{-1}\left[\frac{s}{s^2-4}\right] + L^{-1}\left[\frac{2}{s^2-4}\right]$

$= 3\cosh 2t + \sinh 2t$

I am write at last

⑥

given $L\{e^{3t}\sin 2t\}$

where $\sin 2t = \frac{a}{s^2+a^2}$

$= \frac{2}{s^2+4}$

according to shifting theorem

$\{e^{3t}\sin 2t\} = \frac{2}{(s-3)^2+4}$

$= \frac{2}{s^2+9-6s+4}$ //

$= \frac{2}{s^2-6s+13}$ //

7) (a) given $L^{-1}\left[\frac{s^2}{(s^2+25)^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+25}\right] \cdot L^{-1}\left[\frac{s}{s^2+25}\right]$

$\cancel{f(t) * g(t)}$

$= \cos 5t \cdot \cos 5t$

$f(t) * g(t)$

where as the formula

$\boxed{L\{f(t) * g(t)\} = \int_0^t f(u) \cdot g(t-u)\, du}$

$= \int_0^t \left[\cos 5u \cdot \cos 5(t-u)\right] du$

$= \int_0^t \left[\cos 5u \cdot \cos (5t-5u)\right] du$

$\cos A \cdot \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$

$= \int_0^t$

$= \frac{1}{2}\int_0^t \left\{\cos(5u+5t-5u) + \cos(5u-5t+5u)\right\} du$

$= \frac{1}{2}\int_0^t \left[\cos 5t + \cos(10u-5t)\right] du$

$= \frac{1}{2}\left[\cos 5t\,(u)_0^t + \frac{\sin(10u-5t)}{10}\Big]_0^t\right]$

$= \frac{1}{2}\left(t\cos 5t + \frac{1}{10}\sin(10u-5t)\right]_0^t$

$= \frac{1}{2}\left[t\cos 5t + \frac{1}{10}\left\{\sin[10t-5t] - \sin[10(0)-5t]\right\}\right]$

$= \frac{1}{2}\left[t\cos 5t + \frac{1}{10}\sin 5t + \frac{1}{10}\sin 5t\right]$

$= \frac{1}{2}\left[t\cos 5t + \frac{1}{10}\, 2\sin 5t\right]$

$= \frac{1}{2}\left[t\cos 5t + \frac{\sin 5t}{5}\right]$

(5) (a) given $L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right]$

these is type - I then

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-9}$$

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A(s+4)(s-9) + B\,s(s-9) + C\,s(s+4)}{s(s+4)(s-9)} \rightarrow (1)$$

$$s+2 = A(s+4)(s-9) + B\,s(s-9) + C\,s(s+4) \rightarrow (2)$$

substitute

$s+4=0$

$\boxed{s=-4}$

in above equation

$-4+2 = B(-4)(-4-9)$

$-2 = B\,-4(-13)$

$-2 = B(52)$

$B = \frac{-2}{52}$ then $\boxed{B = \frac{-1}{26}}$

substitute $\boxed{s=0}$

$2 = A(4)(-9)$

$2 = 36A$

$A = \frac{2}{36}$

$\boxed{A = \frac{1}{18}}$

then $s=9$

$9+2 = C\,9(9+4)$

$11 = C\,9(13)$

$11 = 117C$

$\boxed{C = \frac{11}{117}}$

then we substitute in equation (1)

$$L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right] = \frac{\frac{1}{18}}{s} + \frac{\frac{-1}{26}}{(s+4)} + \frac{\frac{11}{117}}{s-9}$$

$$= \frac{1}{18}\left(\frac{1}{s}\right) - \frac{1}{26}\left(\frac{1}{s+4}\right) + \frac{11}{117}\left(\frac{1}{s-9}\right)$$

$$= \frac{1}{18}(1) - \frac{1}{26}e^{-4t} + \frac{11}{117}e^{9t}$$

$$= \ldots e^{-4t} \ldots e^{9t}$$

given gauss divergence

$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

formula for gauss divergence

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^c\int_0^b \left(\frac{x^2}{2} + xy + xz\right)_0^a dy\, dz$$

$$= 2\int_0^c\int_0^b \left(\frac{a^2}{2} + ay + az\right) dy\, dz$$

$$= \cancel{2}\int_0^c \left[\frac{a^2 y + ay^2 + 2ayz}{\cancel{2}}\right]_0^b dz$$

$$= \int_0^c \left[a^2y + ay^2 + 2ayz\right]_0^b dz$$

$$= \int_0^c (a^2b + ab^2 + 2abz)\, dz$$

$$= \left[a^2bz + ab^2z + \frac{2abz^2}{2}\right]_0^c$$

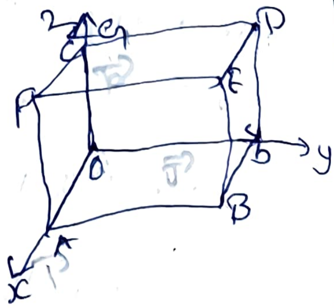
$$= a^2bc + ab^2c + abc^2$$

$$= abc\,[a+b+c] \quad \text{R.H.S}$$

R.H.S $= abc(a+b+c)$

then L.H.S

$= \iint_S \vec{F}\cdot\vec{n}\,ds$



Face	equation	outward normal $\vec{n}$	$\vec{F}\cdot\vec{n}$	ds
ABEF	$x=a$	$\vec{i}$	$a^2-yz=0$ $a^2-yz=0$	$dy\,dz$
OCDG	$x=0$	$-\vec{i}$	$0-(yz)(-1)$ $=yz$	$dy\,dz$
BCEF	$y=b$	$\vec{j}$	$b^2-zx=0$	$dz\,dx$
OCPG	$y=0$	$-\vec{j}$	$0-zx(-1)$ $=zx$	$dz\,dx$
OGFA	$z=c$	$\vec{k}$	$c^2-xy=0$ $=-xy$	$dx\,dy$
OABC	$z=0$	$-\vec{k}$	$0-xy(-1)$ $=xy$	$dx\,dy$

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_S \vec{F}\cdot\vec{n}\,ds + \iint \vec{F}\cdot\vec{n}\,ds + \iint \vec{F}\cdot\vec{n}\,ds + \iint \vec{F}\cdot\vec{n}\,ds$$
$$+ \iint \vec{F}\cdot\vec{n}\,ds + \iint \vec{F}\cdot\vec{n}\,ds$$

(i) $\iint_S \vec{F}\cdot\vec{n}\,ds = \iint a^2-yz\,dy\,dz$

$= \int_0^c\int_0^b a^2-yz\,dy\,dz$

$= \int_0^c \left[a^2y-\frac{y^2z}{2}\right]_0^b dz$

$= \int_0^c \left(a^2b-\frac{b^2z}{2}\right)dz$

$= \left(a^2bz-\frac{b^2z^2}{4}\right)_0^c$

$= a^2bc-\frac{b^2c^2}{4}$ //

b) 2) $= \iint yz\, dy\, dz$

$= \int_0^c \int_0^b yz\, dy\, dz$

$= \int_0^c \left[\frac{y^2 z}{2}\right]_0^b dz$

$= \int_0^c \frac{b^2 z}{2}\, dz$

$= \left[\frac{b^2 z^2}{4}\right]_0^c$

$= \frac{b^2 c^2}{4}$

3) $= \iint b^2 - zx\, dx\, dz$

$= \int_0^c \int_0^a b^2 - zx\, dx\, dz$

$= \int_0^c \left[b^2 x - \frac{z x^2}{2}\right]_0^a dz$

$= \int_0^c b^2 a - \frac{z a^2}{2}\, dz$

$= \left[b^2 a z - \frac{z^2 a^2}{4}\right]_0^c$

$= b^2 ac - \frac{c^2 a^2}{4}$

4) $\iint zx\, dz\, dx$

$= \int_0^c \int_0^a zx\, dx\, dz$

$= \int_0^c \left(\frac{z x^2}{2}\right)_0^a dz$

$= \int_0^c \left(\frac{z a^2}{2}\right) dz$

$= \left(\frac{z^2 a^2}{4}\right)_0^c$

$= \frac{c^2 a^2}{4}$

5) $\iint c^2 - xy\, dx\, dy$

$= \int_0^b \int_0^a c^2 - xy\, dx\, dy$

$= \int_0^b \left[c^2 x - \frac{x^2 y}{2}\right]_0^a dy$

$= \int_0^b \left[c^2 a - \frac{a^2 y}{2}\right] dy$

$= \left[c^2 a y - \frac{a^2 y^2}{4}\right]_0^b$

$= c^2 ab - \frac{a^2 b^2}{4}$

6) $= \int_0^b \int_0^a xy\, dx\, dy$

$= \int_0^b \left(\frac{x^2 y}{2}\right)_0^a dy$

$= \int_0^b \left(\frac{a^2 y}{2}\right) dy$

$= \left(\frac{a^2 y^2}{4}\right)_0^b$

$= a^2 b^2 / 4$

$\iint_S \vec{F}\cdot\vec{n}\,ds = \iint_S \vec{F}\cdot\vec{n}\,ds + \iint_S \vec{F}\cdot\vec{n}\,ds + \iint_S \vec{F}\cdot\vec{n}\,ds + \iint_S \vec{F}\cdot\vec{n}\,ds$
$+ \iint_S \vec{F}\cdot\vec{n}\,ds + \iint_S \vec{F}\cdot\vec{n}\,ds$

$= a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + b^2ac - \cancel{\frac{c^2a^2}{4}} + \cancel{\frac{c^2a^2}{4}} + c^2bc - \cancel{\frac{a^2b^2}{4}}$
$+ \cancel{\frac{a^2b^2}{4}}$

$= a^2bc + b^2ac + c^2bc$

$= abc(a+b+c)$

therefore L.H.S = R.H.S

(6) (5) given $L(t^2\cosh 3t)$

$L\{t^2\cosh at\} = (-1)^2 \frac{d^n}{ds^n}\left[\frac{s}{s^2-9}\right]$

$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2-9}\right]$

$= \frac{d^2}{ds^2}\left[\frac{s}{s^2-9}\right]$

$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}}$

$= \frac{d}{ds}\left[\frac{s^2-9(1) - s(2s)}{(s^2-9)^2}\right]$

$= \frac{d}{ds}\left(\frac{s^2-9-2s^2}{(s^2-9)^2}\right)$

$= \frac{d}{ds}\left[\frac{-s^2-9}{(s^2-9)^2}\right]$

$= -\frac{d}{ds}\left(\frac{s^2+9}{(s^2-9)^2}\right)$

23/12

(b) $= \frac{d^2}{ds^2}\left\{\frac{s}{s^2-9}\right\}$

$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}}$

$= \frac{d}{ds}\left[\frac{s^2-9\,(1) - s\,(2s)}{(s^2-9)^2}\right]$

$= \frac{d}{ds}\left[\frac{s^2-9-2s^2}{(s^2-9)^2}\right]$

$= \frac{d}{ds}\left[\frac{-s^2-9}{(s^2-9)^2}\right]$

$\boxed{\frac{d}{ds}\frac{u}{v} = \frac{vdu-udv}{v^2}}$

$= -\frac{d}{ds}\left(\frac{s^2+9}{(s^2-9)^2}\right]$

$= \frac{-d}{ds}\left[\frac{(\cancel{s^2}-9)^2(2s) - (s^2+9)\,2s\,(s^2-9)(2s)}{((s^2-9)^2)^2}\right]$

$= \frac{-d}{ds}\left[\frac{(s^4+81-18s^2)(2s) - (s^2+9)\,2s\,(s^2-9)}{((s^2-9))^4}\right]$

$\frac{18}{18+2}$

$\frac{81\times 2s}{162}$

$= \frac{-d}{ds}\left(\frac{\cancel{s^4+81-2s^2}\ 2s^5+162s-36s^3 - (s^2+9)(2s^3-18s)}{(s^2-9)^4}\right)$

$= \frac{-d}{ds}\left(\frac{2s^5+162s-36s^3-(2s^5-18s^3+18s^3-162s)}{(s^2-9)^4}\right)$

(s^2+9)

$\frac{101}{102}$ 324

$= \frac{-d}{ds}\left[\frac{2s^5+162s-36s^3-2s^5+18s^3+162s}{(s^2-9)^4}\right]$

$$= \frac{-d}{ds}\left[\frac{\cancel{162}\ 162s\ \cancel{36}s^3 + 162s^4)}{(s^2-9)^4}\right.$$

$$\begin{array}{r} 162 \\ 162 \\ \hline 324 \end{array}$$

$$= \frac{-d}{ds}\left[\frac{324s\ \cancel{36}s^3}{(s^2-9)^4}\right]$$

$$= \frac{d}{ds}\left(\frac{\cancel{36}s^3 - 324s}{(s^2-9)^4}\right)$$

1. Gauss Divergen theorem ?

$$\int_C P\,dx + Q\,dy = \iint_R \left[\frac{d\phi}{dx} - \frac{d\varphi}{dy}\right] dx\,dy$$

$\boxed{\text{where } R = \text{Region}}$

Green's thm

4. given $L\{\sin 5t\ \cos 3t\}$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$= L\{\sin(5t+3t) + \sin(5t-3t)\}$$

$$= L\{\sin 8t + \sin 2t\}$$

$$= L\{\sin 8t\} + L\{\sin 2t\}$$

$$L\{\sin 5t\ \cos 3t\} = \frac{8}{s^2+64} + \frac{2}{s^2+4}$$

K. Surendra Babu.

Sec ÷ G1

Roll No ÷ 40 + 17

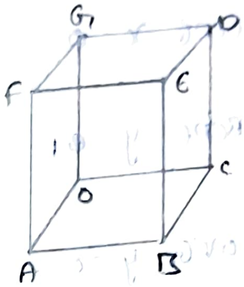
A.No ÷ 7712

Roll No ÷ U22EC174.

SET-A

PART-C

18/30



8b) $\iint_S \vec{F}\cdot\vec{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv.$

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$

~~R.H.S~~ $= 4z - 2y + y$

$\nabla\vec{F} = 4z - y$

$R.H.S = \int_0^1\int_0^1\int_0^1 [4z - y]\, dx\, dy\, dz$

$= \int_0^1\int_0^1 [4xz - xy]_0^1\, dy\, dz$

$= \int_0^1\int_0^1 [4z - y]\, dy\, dz$

$= \int_0^1 \left(4yz - \frac{y^2}{2}\right]_0^1 dz$

$= \int_0^1 \left(4z - \frac{1}{2}\right] dz$

$= \left[\frac{4z^2}{2} - \frac{z}{2}\right]_0^1$

$= \left[2 - \frac{1}{2}\right]$

$= \left[\frac{4-1}{2}\right] = 3/2$

//.

$L.H.S = \iint_S \vec{F}\cdot\vec{n}\, ds.$

face	equation	outward Normal	$\vec{F}\cdot\vec{n}$	ds.
$S_1 = ABEF$	$x = 1$	$-\vec{i}$	$4xz = 4z$	$dy\,dz$
$S_2 = OCDG$	$x = 0$	$-\vec{i}$	$4xz = 0$	$dy\,dz$
$S_3 = BCDE$	$y = 1$	$\vec{j}$	$-y^2 = -1$	$dz\,dx$
$S_4 = OAFG$	$y = 0$	$-\vec{j}$	$-y^2 = 0$	$dz\,dx$
$S_5 = EDFG$	$z = 1$	$\vec{k}$	$yz = y$	$dx\,dy$
$S_6 = OABC$	$z = 0$	$-\vec{k}$	$yz = 0$	$dx\,dy$.

$\iint_S \vec{F}\cdot\vec{n}\, ds = S_1 = \vec{F}\cdot\vec{n}\, ds + S_2 = \vec{F}\cdot\vec{n}\, ds.\ S_3 = \vec{F}\cdot\vec{n}\, ds$

$S_4 = \vec{F}\cdot\vec{n}\, ds.\ S_5 = \vec{F}\cdot\vec{n}\, ds.\ S_6 = \vec{F}\cdot\vec{n}\, ds.$

$$S_1 = \int_0^1\int_0^1 (4z)\, dy\, dz$$

$$= \int_0^1 [4yz]_0^1\, dz$$

$$= \int_0^1 (4z)\, dz$$

$$= \left(\frac{4z^2}{2}\right)_0^1$$

$$= \left(\frac{4}{2}\right)$$

$$= (2)$$

$$S_5 = \int_0^1\int_0^1 (y)\, dx\, dy$$

$$= \int_0^1 [xy]_0^1\, dy$$

$$= \int_0^1 (y)\, dy$$

$$= \left(\frac{y^2}{2}\right)_0^1$$

$$= \left(\frac{1}{2}\right)$$

$$S_3 = -\int_0^1\int_0^1 dz\, dx$$

$$= -\int_0^1 [x]_0^1\, dz$$

$$= -\int_0^1 (1)\, dz$$

$$= -(z)_0^1 = -1$$

$S_1 + S_1 + S$

$S_1 + S_1 + S_5$

$= 2 + (-1) + \frac{1}{2}$

$= \frac{4 - 2 + 1}{2}$

$= \frac{5-2}{2}$

$= 3/2$

10

Sub eq ① & ②

L.H.S = R.H.S.

PART - B

6) (b) $L\left[t^2 \cos 2t\right]$

Sol:- $L\left[t^2 e^{-3t} \cos 2t\right] = (-1)^2 \frac{d^n}{ds^n} f(s)$

$L(\cos 2t) = (-1)^2\left[\frac{s}{s^2+4}\right] = f(s)$

$L\left(t^2 \cos 2t\right) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$

$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$

$\left(\frac{d}{dx} \frac{u}{v} = \frac{vdu - udv}{v^2}\right)$

$= \frac{d}{ds} \frac{(s^2+4) \cdot 1 - s(s^2+4)\, 2s}{((s^2+4)^2)}$

3

$= \frac{d}{ds} \frac{(s^2+4)^2 \cdot 2s}{(s^2+4)^2}$

$= \frac{d}{ds} \frac{(s^2+4)^2 \cdot \cancel{8} - (s^2+4) \cdot 2s}{((s^2+4)^2)^2}$

$$\frac{d}{ds} = \left[\frac{(s^2+4)-2s}{(s^2+4)^{4}}\right]_{s\to s-3}$$

$$\frac{d}{ds} = \frac{-2s^3-16s-8s}{(s^2+4)^3}$$

$$\frac{d}{ds} = \frac{-2s^3-24s}{(s^2+4)^3}$$

$$\frac{d}{ds} = \left[\frac{-2s(s^2-12)}{(s^2+4)^3}\right]$$

7) a) Sol:-

$$L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right]$$

$$=\left[\frac{s+2}{s(s+4)(s-9)}\right] = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+9)}$$

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A(s+4)(s+9)+B(s)(s+9)+C(s)(s+4)}{s(s+4)(s+9)}$$

$$s+2 = A(s+4)(s+9) + B(s)(s+9) + C(s)(s+4)$$

Put $s = -4$

$$-4+2 = A(-4+4)(4+9) + B(-4)(-4+9) + C(-4)(-4+4)$$

$$-2 = A(0) - B(-4)(-13) + C(0)$$

$$-2 = B\,52$$

$$B = \frac{-2}{52} = \frac{1}{26}$$

$$\boxed{B = \frac{1}{26}}$$

Put $s = -9$

Put $s=0$. $A(4)(0-9)$

$2 = A(4)-9$

$2 = -36A$

$A = -\frac{1}{18}$

$s=9 = -11-9 \quad C(9+4)$

$11 = 117C$

$\boxed{C = \frac{11}{117}}$

$L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right] = \cancel{\frac{\frac{1}{26}}{s}} L^{-1}\left(\frac{\frac{1}{26}}{s} + \frac{\frac{1}{18}}{(s+4)} + \frac{\frac{11}{117}}{(s-9)}\right)$

$= \cancel{\left(L^{-1}\frac{1}{26}\left(\frac{1}{s}\right) + L^{-1}\frac{1}{18}\frac{1}{(s+4)} + \frac{11}{117}\frac{1}{(s-9)}\right)}$

$= \cancel{L^{-1} e^{t}\frac{1}{26}}$

$= L^{-1}\frac{\frac{1}{18}}{(s)} + L^{-1}\frac{\frac{1}{26}}{(s+4)} + L^{-1}\frac{\frac{11}{117}}{(s-9)}$

$= L^{-1}\frac{1}{18}e^{t} - L^{-1}e^{-4t}\frac{1}{26} + L^{-1}e^{9t}\frac{11}{117}$

$= L^{-1}e^{t}\frac{1}{18} - L^{-1}e^{-4t}\frac{1}{26} + L^{-1}e^{9t}\frac{11}{117}$

3

Name:- J. Sisindar Reddy.

Register No:- U22EC165.

section :- G1 ECE

Subject code:- U20MABT02.

11/30

7/a. $L^{-1}\left[\frac{s+2}{s(s+4)(s-9)}\right]$.

consider as.

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-9}.$$

$$\frac{s+2}{s(s+4)(s-9)} = \frac{A(s+4)(s-9) + B(s)(s-9) + C(s)(s+4)}{s(s+4)(s-9)}.$$

$$s+2 = A(s+4)(s-9) + B(s)(s-9) + C(s)(s+4) \quad -①$$

put $s=-4$.

$A(-4+4)(-4-9)$	$B(-4)(-4-9)$	$C(-4)(-4-4)$
$A(0)(-13)$	$B(-4)(-13)$	$C(-4)(0)$
$A(0)$.	$B(52)$.	$C(0)$.

$2 = B(52)$

$B = \frac{-\cancel{2}^{1}}{\cancel{52}_{26}}$

$B = \frac{-1}{26}$. $= B = \frac{-1}{26}$.

put $s=0$.

$A(0+4)(0-9)$	$B(+0)(0-9)$	$C(0)(0-4)$
$A(4)(-9)$	$B(0)$	$C(0)$.
$A(-36)$		

$2 = A(-36)$

$A = \frac{\cancel{2}^{1}}{-\cancel{36}}. = A = \frac{-1}{18}.$

Put $s = 9$.

$A(+9+4)(9-9)$ | $B(+9)(9-9)$ | $C(9)(9+4)$

$A(+13)(0)$ | $B(9)(0)$ | $C(9)(13)$

$A(0)$. | $B(0)$. | $C(117)$.

$9+2 = C(117)$

$11 = C(117)$

$C = \frac{11}{117}$.

$$\frac{s+2}{s(s+4)(s-9)} = \frac{\frac{-1}{18}}{s} - \frac{\frac{1}{26}}{s+4} + \frac{\frac{11}{117}}{s-9}.$$

$$L^{-1}\left\{\frac{s+2}{s(s+4)(s-9)}\right\} = -\frac{1}{26}L^{-1}\left(\frac{1}{s+4}\right) + \frac{11}{117}L^{-1}\left(\frac{1}{s-9}\right) - \frac{1}{18}\left(\frac{1}{s}\right)$$

$$L^{-1}\left\{\frac{s+2}{s(s+4)(s-9)}\right\} = -\frac{1}{18} \cdot 1 - \frac{1}{26}e^{-4} + \frac{11}{117}e^{at}.$$

02.
Sol).

$\vec{F} \cdot x^2\vec{i} + y^2\vec{j}$ and $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

$\int \vec{F} \cdot d\vec{r} = \int (x^2 dx + y^2 dy)$

$\int \vec{F} \cdot d\vec{r} = \int x^2 dx + x^2 dy$

$= \int 2x^2 dx$.

$= 2\left(\frac{x^3}{3}\right)$

$= \frac{2x^3}{3}$ //.

8/6.

$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

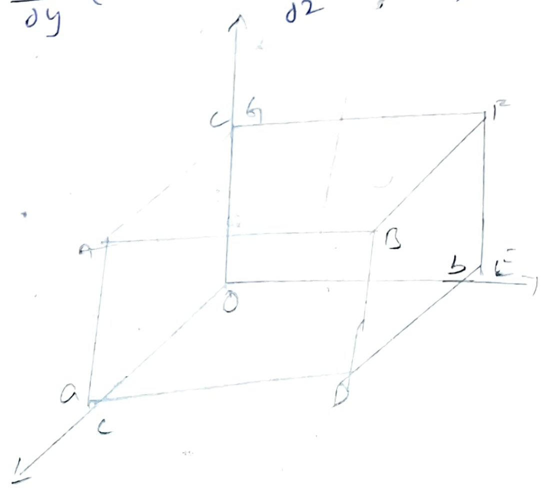
$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

$\nabla \cdot = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$\vec{F} \cdot = (x + x\vec{i}y) + (y + xz) + (z + xy)$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x + xy) + \frac{\partial}{\partial y}(y + xz) + \frac{\partial}{\partial z}(z + xy)$

$\int_0^c \int_0^a \int_c^b$



$\frac{B}{9}$, $L(e^{-3t}\sinh 2t)$.

$L[e^{-at} f(t)] = f(s+a)$

$f(t) = \sinh 2t$

$= \frac{2}{s^2+4}$.

$L(e^{-3t}\sinh 2t) = \left(\frac{2}{s^2+4}\right)_{s \to s+3}$.

$= \frac{2}{(s+3)^2+4}$

4

$= \frac{2}{s^2+9+6s+4}$

$= \frac{2}{s^2+6s+13}$.

ii, $L[f'(t)] = L'(f(t)) - L(f(0))]$

iii $L(f''(t)) = L[f''(t) - L(f'(0)) - L(f(0))]$

CLA-3

Name - Neha Jana
Section - A1
REG. NO. - U22AE022
Sub - ACCA (U20MABT02)

CLA-3

30

30 - good

①

Set-A

1) $w = \frac{6z-9}{z}$

Replace z in the place of w

$z = \frac{6z-9}{z}$

$z^2 = 6z - 9$

$z^2 - 6z + 9 = 0$

$z^2 - (3+3)z + 9 = 0$

$z^2 - 3z - 3z + 9 = 0$

$z(z-3) - 3(z-3) = 0$

$(z-3)(z-3) = 0$

$z = 3, 3$

∴ The fixed point of the transformation is 3.

2) Cauchy's Integral formula for Derivatives –

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz$$

$$f'(z_0) = \frac{1!}{2\pi i} \int \frac{f(z)}{(z - z_0)^2} dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \int \frac{f(z)}{(z - z_0)^3} dz$$

$$f^n(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z - z_0)^{n+1}} dz$$

3) $f(z) = \frac{1}{z+i}$

The singular point of the function is $-i$.

4) $f(z) = \frac{z}{(z+1)(z+2)}$

In this equation singular point is

$z = -1$ and $z = -2$

$$\lim_{z \to -2} (z+2) \cdot \frac{z}{(z+1)(z+2)}$$

$$= \frac{-2}{(-2+1)} = \frac{-2}{-1} = +2$$

5) $f(z) = \cot z$

$$f(z) = \frac{\cos z}{\sin z}$$

$$= \frac{\cos z}{\cos z} = 1$$

the residue at $z=0$ for $f(z) = \cot z$ is 1

Part-B

6 b) Bilinear transformation that maps the point $z = 0, -i, -1$ onto the points $w = i, 1, 0$ respectively.

Soln. w.k.t Bilinear transformation is

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_2 - w_3)(w_3)}$$

$w_1 = i$
$w_2 = 1$
$w_3 = 0$
$z_1 = 0$
$z_2 = -i$
$z_3 = -1$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\frac{1\cdot w-i)}{-wi+w} = \frac{z(-i+1)}{i(-1-z)}$$

$$\frac{w-i}{wi+w} = \frac{-zi+z}{-i+zi}$$

$$(w-i)(-i-zi) = (wi+w)(-zi+z)$$

$$-wi-wzi+(i^2)+zi^2 = wzi^2+wiz-wzi+wz$$

$$-wi-wzi-1+zi^2 = -\cancel{w}z+\cancel{w}iz-\cancel{w}zi+\cancel{w}z$$

$$-wi-wzi-1-z=0$$

$$i(w\cancel{-}wz) = z+1$$

$$w(i-zi) = z+1$$

$$w = \frac{z+1}{i-zi}$$

7.a) Evaluate $f(z) = \cos z$ as a Taylor's series about the point $z = 0$

Soln.

w.k.t Taylor's series Formula is —

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) \dots +$$

$$f(z) = \cos z$$

Function	Value of $z=0$
$f(z) = \cos z$	$f(0) = \cos(0) = 1$
$f'(z) = -\sin z$	$f'(0) = -\sin(0) = 0$
$f''(z) = -\cos z$	$f''(0) = -\cos(0) = -1$
$f'''(z) = \sin z$	$f'''(0) = \sin(0) = 0$
$f^{iv}(z) = \cos z$	$f^{iv}(0) = \cos(0) = 1$

$$f(z) = f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0) + \frac{(z-0)^4}{4!} f^{iv}(0)$$

$$= 1 + 0 + \frac{z^2}{2}(-1) + \frac{z^3}{3!}(0) + \frac{z^4}{4!}(1)$$

$$= 1 - \frac{z^2}{2} + \frac{z^4}{24}$$

PART-C

8.b) Find the image of $|z-2i|=2$ under the transformation $w = 1/z$.

Soln.

$$w = \frac{1}{z}$$

W.K.T Cauchy's Integral Formula

$$\int \frac{f(z)}{z-z_0} = 2\pi i f(z_0)$$

Here, $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$z = \frac{\bar{w}}{w\bar{w}}$$

$$z = \frac{\bar{w}}{|w|^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

Equating Real Part and Imaginary part,

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Here given, $|z-2i|=2$

$$|x+iy-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$x^2+y^2-4y+4 = 4$$

$$x^2+y^2-4y = 0$$

$$x^2+y^2-4y=0$$

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-4\left(\frac{-v}{u^2+v^2}\right)=0$$

$$\frac{u^2}{(u^2+v^2)}+\frac{v^2}{u^2+v^2}+\frac{4v}{u^2+v^2}=0$$

$$\frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}=0$$

12

$$\frac{1}{u^2+v^2}+\frac{4v}{u^2+v^2}=0$$

$$1+4v=0$$

the st. line $1+4v$ is in the w-plane.

6.a) Determine the Analytic function $f(z)$, whose real part is $x^3-3xy^2+3x^2-3y^2+1$.

Soln.

Here given the real part is

$$u=x^3-3xy^2+3x^2-3y^2+1$$

$$u_x=3x^2-3y^2+6x$$

$$u_y=-6xy-6y$$

Replace the value in the place of $x=z$ and $y=0$

$$\phi_1(z,0)=3z^2-3(0)+6z=3z^2+6z$$

$$\phi_2(z,0)=-6(z).0-6.0=0$$

$\therefore f(z)$

$$f(z) = \int [\phi_1(z,0) - i\phi_1(z,0)]\, dz$$

$$= \int (3z^2 + 6z)\, dz$$

$$= 3 \cdot \left[\frac{z^3}{3}\right] + 6\left[\frac{z^2}{2}\right]$$

$$= z^3 + 3z^2$$

PART-A

1) $f(z) = \dfrac{z}{(z+1)(z+2)}$

The singular points of the function is
$z = -1$ and $z = -2$

$$\lim_{z \to -1} (z+1) \frac{z}{(z+1)(z+2)}$$

$$= \frac{-1}{-1+1}$$

Given Integration

$$f(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{z/(z+1)}{\frac{(z+1)(z+2)}{(z+1)}} = \frac{z/(z+1)}{(z+2)}$$

$f(z) = z/(z+1)$, $z_0 = -2$

$$f(z_0) = \frac{z_0}{z_0+1}$$

$$f(-2) = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

$$\int \frac{f(z)}{z - z_0} = 2\pi i\, f(-2)$$

$$= 4\pi i$$

8(a) Find $f(z)$, if the Imaginary part is $e^x(x\sin y + y\cos y)$. Also find it conjugate.

soln.

Here given the imaginary part is

$$v = e^x(x\sin y + y\cos y)$$

$$v_y = e^x.x(+\cos y) + e^x(y(-\sin y) + \cos y))$$

$$\phi_1(z,0) = e^z.z(+1) + e^z$$

$$= e^z z + e^z = e^z(z+1)$$

$$v_x = \sin y(e^x + e^x x) + e^x y\cos y$$

$$\phi_2(z0) = 0$$

w.k.t Milton Thomas Method,

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\,dz$$

$$= \int e^z(z+1)\,dz$$

$$= ze^z$$

$$f(z) = ze^z$$

$$u + iv = (x+iy)e^{x+iy}$$

$$= (x+iy).e^x e^{iy}$$

$$= (xe^x + iye^x)e^{iy}$$

To find its conjugate

~~$u+iv = (xe^x + iye^x$~~

$$f(z) = \cancel{(x+iy)}\,(xe^x + iye^x).(\cos y + i\sin y)$$

~~$f(z) =$~~

$$f(z) = (xe^x + iye^x)(\cos y + i\sin y)$$
$$= xe^x\cos y + xe^x i\sin y + iye^x\cos y + i^2 ye^x\sin y$$
$$= xe^x\cos y + xe^x i\sin y + iye^x\cos y - ye^x\sin y$$
$$= (xe^x\cos y - ye^x\sin y) + i(xe^x\sin y + ye^x\cos y)$$

Name : G. Arun kumar

Sec : A1

Reg No : U22AE007

Sub : Advanced Calculus and Complex Analysis

Sub Code : U20MABT02

Set : A

Maths - CLA - III.

Part - B.

19/30

6 b. Given points are

$z = 0, -i, -1$

$w = i, 1, 0$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad \text{--- (1)}$$

z and w points, sub in (1)

$$\frac{(w-i)\cancel{(1-0)}}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\frac{(w-i)}{(i-wi)} = \frac{(-z-1)}{(i-z)}$$

$(w-i)(i-z) = (i-wi)(-z-i)$

$wi - zwi + 1 - zi = -zi + 1 + zwi - wi$

$$w = \frac{\cancel{1} i(1-z)}{\cancel{1}(z-i)}$$

4

$$w = \frac{i(1-z)}{(z-i)}$$ //

7 a.

$$f(a) + \frac{(z-a)}{1!} f(a)^2 + \frac{(z-a)^2}{2!} f(a)^3 + \frac{(z-a)^3}{3!}$$

function	Value at $z=0$
$f(z) = \sin(z)$	$f(0) = \sin(0) = 0$
$f(z) = -\sin(z)$	$f(0) = -\sin(0) = 0$
$f(z) = \cos(z)$	$f(0) = \cos(0) = 1$
$f(z) = -\cos(z)$	$f(0) = -\cos(0) = -1$

$z = 0$

$$f(a) + \frac{(z-a)}{1!} f(a) + \frac{(z-a)^2}{2!} f(a) + \frac{(z-a)^3}{3!}$$

$$0 + \cancel{\frac{0-a}{1!}}\ 1 + \frac{(0-a)^2}{2!}\ 0 + \frac{(0-a)^3}{3!}$$

$$0 + \frac{0-a}{1!}\ 1 + \frac{(0-a)^2}{2!}\ 0 + \frac{(0-a)^3}{3!} \ldots$$

Part - C

8 b. $|z-2i| = 2$

$w = 1/z$

$z = \frac{1}{w}$

$z = \frac{\bar{w}}{w.\bar{w}}$

$z = \frac{w}{(w)^2}$

$z = x+iy$, $w = u+iv$

$x+iy = \frac{u-iv}{(u+iv)^2}$

$x+iy = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

Then,

$\cancel{(z-2i) = 2}$

$|(x+iy-2i)| = 2$

$|x+i(y-2)| = 2$

Squaring on both sides

$(x^2 + (y-2)^2 = 2$

$= x^2+y^2-4y+\cancel{4} = \cancel{4}$

$= x^2+y^2-4y = 0$

$$= \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$= \frac{u^2+v^2}{(u^2+v^2)^2} + 4\left(\frac{v}{u^2+v^2}\right) = 0$$

$$= \frac{1}{u^2+v^2} + \frac{4v}{u^2+v^2} = 0$$

10

$$= \boxed{1+4v=0}$$

Part - A

3 Singular point of the function

$$f(z) = \frac{1}{z+i}$$

Given $f(z) = \frac{1}{z+i}$

Here

$i = -i$ is the singular point for given function $f(z) = \frac{1}{z+i}$

2
$$\int_C \frac{f(x)}{z-z_0}\, dx = 2\pi i\, f(z_0)$$

4 $f(x) = \frac{x+1}{(z+2)(z+3)}$, $x = -2$

$= -2xi.$

BIHER

CLA-3

Name :- Saurabh Kumar Poddar
Registration No:- U22AS024
Admission No.:- 9144
Subject :- U20MABT02 - ADVANCED CALCULUS AND COMPLEX ANALYSIS

Set :- B

CLA :- 3rd

Date :- 31.01.2023

section :- A1

31/01/2023

8/30

Q.1.) Check whether the function $f(z) = z^2$ is analytic or not?

Sol^n.

$f(z) = z^2$

~~u+iy~~ $u+iv = x+iy$

$u+iv = (x+iy)^2$

$u+iv = x^2 - y^2 + 2ixy$

~~D. w.r.to x & y~~

$u = x^2 - y^2$

$v = 2ixy$

diff. w.r.to x & y

$\frac{\partial u}{\partial x} = 2x \;,\; \frac{\partial v}{\partial x} = 2y$

$\frac{\partial u}{\partial y} = -2y \;,\; \frac{\partial v}{\partial y} = +2x$

C-R Equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;,\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ function is analytic

(6) (b) Soln

$w = 0, i, \infty$

$z = \infty, i, 0$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

$$= \frac{(w)(i-\infty)}{-i(\infty-w)} = \frac{(z-\infty)i}{-z(\infty-i)}$$

$$= \frac{iw-w\infty}{-i\infty+iw} = \frac{iz-i\infty}{-z\infty+iz}$$

$$=(iw-w\infty)(-z\infty+iz) = (iz-i\infty)(-i\infty+iw)$$

$$= -izw\infty + zw\infty^2 + (-zw) - iwz\infty = z\infty - \infty^2 - zw + w\infty$$

BJHER

4. → $\cancel{f(z_0) = \int \frac{f(z)}{z-z_0} dz}$

$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0} dz.$

7.(a) $f(z) = \sin z$

$\cancel{f(0) = \sin 0}$
$\cancel{= 0 \text{ Ans}}$

$f(z) = \cos z$

$f(0) = \cos 0$

$= 1$

$\therefore f(z) = \sin z$

$= 1$ Ans

(b) Given

$$\int_C \frac{z+1}{(z-3)(z-1)} dz$$

Cauchy's integral formula

$f(z) =$

(8) $\frac{\sin 2x}{\cos 2h + \cos 2x}$

$u_x = \frac{\cos h 2y + \cos 2x)2}{(\cos 2y + \cos 2x)^2}$

$\phi(z,0) = \frac{(1+\cos 2z)\cos 2 \cdot 2) \sin 2x / \sin 2x}{1+\cos 2z}$

similar like

CCA

$\cos h 2y +$

$\frac{1+\cos 2z)\cos z \cdot \sin z \cdot (\sin 2z)}{(1+\cos 2z)^2}$

$= 1$

$\phi_2 = 0$

using mil-theorm

$f(z) = \int [\phi_1(z,0) - \phi_2(z,0)]\, dz$

$= \int \left[\frac{2\cos 2z + 2\cos 2z + 2\sin 2z}{(1+\cos 2z)^2}\right] dz$

$= \int \left[\frac{2(1+\cos 2z)}{(1+\cos 2z)^2}\right] dz$

$\int \left(\frac{2}{1+\cos 2z}\right) dz$

$\int \left(\frac{1}{\cos^2 z}\right) dz$

$\int \sec^2 z\, dz$

$f(z) = \tan z$

NAME : R. Rajesh

Reg NO : U22CN1189

Adm NO : 7207

Subject : ADVANCED CALCULUS & COMPLEX ANALYSIS

Subject Code : U20BTBT01.

Section - K

Set - A

25/4/25

PART - A

30/30

V. Sm

5. Given

$$\phi = xy + yz + zx$$

point at $(1,2,0)$

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$D.D = \vec{a} \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

$$\nabla \phi = \vec{i} \cdot \frac{d\phi}{dx} + \vec{j} \frac{d\phi}{dy} + \vec{k} \frac{d\phi}{dz}$$

$$\frac{d\phi}{dx} = y + 0 + z = y + z.$$

$$\frac{d\phi}{dy} = x + z + 0 = x + z$$

$$\frac{d\phi}{dz} = 0 + y + x = y + x$$

$$\nabla \phi = \vec{i}(2+0) + \vec{j}(1+0) + \vec{k}(2+1)$$

$$= 2\vec{i} + \vec{j} + 3\vec{k}$$

$$|\nabla \phi| = \sqrt{4+1+9} = \sqrt{14}$$

$$D.D = (\vec{i} + 2\vec{j} + 2\vec{k}) \frac{(2\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{14}}$$

$$= \frac{2+2+6}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

$$\boxed{D.D = \frac{10}{\sqrt{14}}}$$

4. Given

$$\phi = \log(x^2+y^2+z^2)$$

$$\nabla\phi = \vec{i}\,\frac{d\phi}{dx} + \vec{j}\,\frac{d\phi}{dy} + \vec{k}\,\frac{d\phi}{dz}$$

$$\frac{d\phi}{dx} = \frac{1}{x^2+y^2+z^2}\,2x = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{d\phi}{dy} = \frac{1}{x^2+y^2+z^2}\,2y = \frac{2y}{x^2+y^2+z^2}$$

$$\frac{d\phi}{dz} = \frac{1}{x^2+y^2+z^2}\,2z = \frac{2z}{x^2+y^2+z^2}$$

$$\nabla\phi = \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{1}{x^2+y^2+z^2}\left[\vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)\right]$$

Now

$$r = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^2 = x^2+y^2+z^2$$

$$= \frac{1}{r^2}\left[2r\right]$$

$$= \frac{2r}{r^2}$$

$$\boxed{\nabla\phi = \frac{2r}{r^2}}$$

3. Given

The triple integral is

$$\int_0^1\int_0^2\int_1^2 x^2yz \; dx\,dy\,dz$$

$$= \int_0^1 \int_0^2 \int_1^2 x^2 y z \, dx\, dy\, dz$$

$$= \int_0^1 \int_0^2 \left[\frac{x^3}{3} yz\right]_1^2 dy\, dz$$

$$= \int_0^1 \int_0^2 \frac{1}{3} yz \left[\frac{8}{3} - \frac{1}{3}\right] dy\, dz$$

$$= \int_0^1 \int_0^2 \frac{1}{3} \cdot \frac{7}{3} \left[\frac{y^2}{2}\right]_0^2 z\, dz$$

$$= \int_0^1 \frac{7}{3} z \left[\frac{4}{2} - \frac{0}{2}\right] dz$$

$$= \int_0^1 \frac{7}{3} \cdot 2 \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{14}{3} \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{14}{3} \left[\frac{1}{2} - \frac{0}{2}\right]$$

$$= \frac{14}{3} \left[\frac{1}{2}\right]$$

$$= \frac{14}{6} = \frac{7}{3}$$

$$\boxed{\int_0^1 \int_0^2 \int_1^2 x^2 y z = 7/3}$$

2.

Given

$$\iint_S x^2 y^2 \, dx\, dy$$

And given line is

$x=0,\ x=3,\ y=0,\ y=3$

$$I=\int_0^3\int_0^3 x^2y^2\,dx\,dy$$

$$=\int_0^3\left[\frac{x^3}{3}\right]_0^3 y^2\,dy$$

$$=\int_0^3 y^2\left[\frac{(3)^3}{3}-\frac{(0)^3}{3}\right]dy$$

$$=\int_0^3 y^2\left[\frac{27}{3}-0\right]dy$$

$$=\int_0^3 y^2[9]\,dy$$

$$=9\int_0^3 y^2\,dy$$

$$=9\left[\frac{y^3}{3}\right]_0^3$$

$$=9\left[\frac{27}{3}-\frac{0}{3}\right]$$

$$=9[9-0]$$

$$=81$$

$$\boxed{I=81}$$

$$\boxed{\int_0^3\int_0^3 x^2y^2\,dx\,dy=81}$$

$\int \frac{1}{x} = \log x$

1. Given

$$I = \int_1^a \int_1^b \frac{1}{xy} \, dx \, dy$$

$$= \int_1^a [\log y]_1^b \, dy$$

$$= \int_1^a [\log b - \log 1] \cdot \frac{1}{y} dy$$

$$= \int_1^a [\log b - 0] \frac{1}{y} dy$$

$$= \int_1^a \log b \left(\frac{1}{y}\right) dy$$

$$= \log b \int_1^a \log y$$

$$= \log b \, [\log y]_1^a$$

$$= \log b \, [\log a - \log 1]$$

$$= \log b \, [\log a - 0]$$

$$= \log a \log b$$

$$\therefore \boxed{\int_1^a \int_1^b \frac{1}{xy} \, dy \, dy = \log a \log b}$$

$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(y-z)^4 - 4(z-x)^3$$

$$= 4(1-y)^3 + 0 + 4(z-1)^3$$

$\frac{\partial u}{\partial y}$

$\frac{d\varphi}{d\varphi}$

PART - B

7.

(a) Given

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\nabla r = \frac{dr}{dx}\vec{i} + \frac{dr}{dy}\vec{j} + \frac{dr}{dz}\vec{k}$

$\nabla r^n = n \cdot r^{n-1}\vec{r}$ $\boxed{a^n = a^{n-1}}$

$\nabla r^n = \frac{n \cdot r^{n-1}}{x}\vec{i} + \frac{n \cdot r^{n-1}}{y} + \frac{n \cdot r^{n-1}}{z}$

$\nabla r^n = \frac{1}{x+y+z}\left[n \cdot r^{n-1} + n\, r^{n-2} + n \cdot r^{n-1}\right]$

$\nabla r^n = \frac{1}{\vec{r}}\left[n \cdot r^{n-2}\right]$ $\boxed{(a^m)^n = a^{m-n}}$

$\nabla r^n = n \cdot r^{n-2} \cdot \vec{r}^{-1}$

$\nabla r^n = n \cdot r^{n-2}$

$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is $\nabla r^n = n \cdot r^{n-2}\vec{r}$

Hence proved.

PART - C

8.

(a) Given

double integral is

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

By changing this into Polar coordinate.

Here

$x = r\cos\theta \quad ; \quad y = r\sin\theta$

$\boxed{\vec{x}+\vec{y}=\vec{r}}$

$r(\cos\theta)^2 + (r\sin^2\theta)^2 = x^2+y^2$

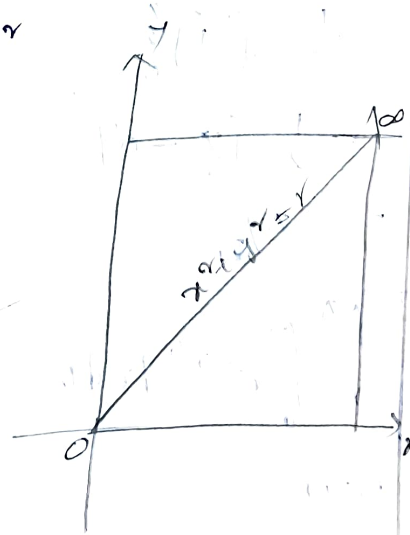
$r^2\cos^2\theta + r^2\sin^2\theta = x^2+y^2$

$\boxed{\cos^2\theta+\sin^2\theta=1}$

$r^2(\cos^2\theta+\sin^2\theta) = x^2+y^2$

$r^2(1) = x^2+y^2$

$\boxed{x^2+y^2=r^2}$



Now

limits are.

r varies from 0 to ∞.

θ varies from 0 to $\frac{\pi}{2}$

Now

Polar coordinate is

$$= \int_0^\pi \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$\boxed{x^2+y^2=r^2}$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, \cancel{dx\,dy}\; dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \, dr \, d\theta$$

$$\boxed{\int e^{-t} \, dt = \frac{e^{-t}}{-1}}$$

put

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \, dr \, d\theta$$

$$\boxed{\begin{aligned} \int e^{-r^2} dt &= dr\,d\theta \\ 2r \, dr\,d\theta &= dt \\ dr\,d\theta &= \tfrac{1}{2} dt \end{aligned}}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot \frac{1}{2} \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{e^{-r}}{-1} \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[e^{-r}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(e^{-\infty} - e^{-0}\right] d\theta$$

$$= \frac{1}{2} \left[1\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0\right]$$

$$= \pi/4$$

$$\boxed{\int_0^{\pi} \int_0^{\infty} e^{-r^2} \, dr \, d\theta = \pi/4}$$

NOW

$$\int_0^{\infty} e^{-x^2} \, dx = \pi/4$$

$$\int_0^{\infty} e^{-x} \, dx = \pi/4$$

$$\left[e^{-x}\right]_0^{\infty} dx = \pi/4$$

$$\left[e^{-\infty} - e^{0}\right] = \pi/4$$

$$\int e^{-x^2} = \pi/4$$

$$r^2 = \pi/4$$

$$r = \sqrt{\pi/4}$$

$$\boxed{r = \sqrt{\pi}/2}$$

$$\boxed{\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

Now

$$\int_0^{\pi}\int_0^{\infty} e^{-r^2} dr\, d\theta = \pi/4$$

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

PART-B

6.
(a) Given

ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Now

the area ~~of a~~ the ellipse is

x varies from 0 to a

y varies from 0 to $y = \pm b\sqrt{\left(1-\frac{x^2}{a^2}\right)}$

Now the limit is

$$= \int_0^a \int_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx\, dy$$

$$= \int_0^a \int_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dy\, dx$$

$$= \int_0^a [y]_0^{b\sqrt{\left(1-\frac{x^2}{a^2}\right)}} dx$$

$$= \int_0^a \left[b\sqrt{\left(1-\frac{x^2}{a^2}\right)} - 0\right] dx$$

$\sqrt{a^2-x^2}$

$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}$

$$= \frac{b}{a^2}\int_0^a \left[\sqrt{a^2-x^2} - 0\right] dx \quad \pm \frac{b}{a^2}$$

$$= \frac{b}{a^2}\int_0^a \left[\frac{a^2}{2}\sin^{-1}(x/a) + \frac{x}{2}\sqrt{a^2-x^2}\right] dx$$

$$= \frac{b}{a^2}\left[\frac{a^2}{2}\sin^{-1}(x/a) + x/2\sqrt{a^2-x^2}\right]_0^a$$

$$= \frac{b}{a^2}\left[\frac{a^2}{2}\sin^{-1}(1) + \frac{a}{2}\sqrt{0-a^2}\right] - \left[\frac{a^2}{2}\sin^{-1}(0) + \frac{0}{2}\sqrt{a^2-0^2}\right]$$

$$= \frac{b}{a^2}\left[\frac{a^2}{2}\sin^{-1}(1) + 0\right] - [0]$$

$$= \frac{b}{a^2}\left[\frac{a^2}{2}\cdot \pi/2 + 0\right] = \frac{a^2 b \pi}{a^2 4}$$

$$= \frac{\pi ab}{4}$$

Area of the ellipse is $\frac{\pi ab}{4}$ //

reg: U22CN177
Date: 25.04.2023/FN
subcode: U20MABT02
Sub: Advanced calculus and complex Analysis
section: K

x/04/2023

SET-A

Part-A

15/30

1. given $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$.

$\int_0^1 x^2 \left(y^2 \int_1^2 \right) dx\,dy$

$\int_0^1 x^2 \int_1^2 y^2 \,(dx\,dy)$

$\int_0^1 dx^3 \int_1^2 dy^3$

the double integral of $\int_0^1 \int_1^2 (x^2+y^2)\,dx\,dy$ is

$\int_0^1 dx^3 \int_1^2 dy^3$

2. given

$$\int_0^1 \int_y^{y^2} (x+y)\,dx\,dy$$

$$\int_0^1 x \int_y^{y^2} y\,(dx+dy)$$

let $y=1$

$$\int_0^1 x \int_1^{1^2} y$$

$$\int x' \int y^2$$

3. given

$$\int_0^{\pi} \int_0^{x} \frac{\sin x}{x}\,dy\,dx$$

the formula is $\sin x = \cos x$, $\cos x = \sin x$

~~π Va~~

let substructe

$x=2$

$\cancel{\sin x}$ $\frac{\sin 2}{2}\,dy\,d2x$

$$\int_0^2 \sin\,(dyd\tfrac{x}{2})$$

$\int_0 \sin^2\theta \, (dy\,dx)$

$\sin^2\theta = dy\,dx$

$dy\,dx = \sin^2\theta$

4. given

$\phi = x^2 + y - z - 1$

the given value are $(1, 0, 0)$

let $x = 1$

let $y = 0$

let $z = 0$

$= 1^2 + 0 - 0 - 1$

$= 1 - 0 - 1$

$= 1 - 1$

$= 0$

the grad $= 0$

5. the given

$\phi = xyz$

$$\begin{pmatrix} i & j & k \\ 1 & 1 & 1 \\ x & y & z \end{pmatrix}$$

~~$i(1z) + j$~~

$i(1z - 1y) + j(1z - 1x) + k(1y - 1x)$

$i(zy) + j(zx) + k(yx)$

$i\cancel{z}y + j\cancel{z}x + kyx$

$iy + jx + kyx$

$i\cancel{y} + j + k\cancel{y}$

$i + j + k$

2

the directional derivative of $\phi = ijk$

PaRT-B

6

b) given

$y = 2x+3$

$y = x^2$

$y = 2x+3$

$= 5x$

$-5 = x$

$x = -5$

$y = -5^2$

$y = 25$

now adding

2 $2(25)+3$

$50+3$

53

the area using double integral bounded is 53

7.

b) The given

$x^2-y^2+z=2$ at the Point $(1,-1,2)$

1^2+1^2+2

$1+1+2$

4 $\rightarrow$ ①

now

$x=2$

$y=2$

$z=3$

2^2-2^2+3

$4+4+3$

$8+3$

11 $\rightarrow$ ②

now

$x = \cancel{3}1$

$y = 2$

$z = 3$

$1^2 - 2^2 + 3$

$1 - 4 + 3$

$-3 + 3$

$0 \rightarrow 3$

121
16
137

now subsutract

$x = 4$

$y = 11$

$z = 0$

$4^2 \cancel{+} -11^2 + 0 =$

$16 + 121 + 0 = 137$

the unit Normal vector to the surface is 137

Part-C

8

b) given

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} \, dx\,dy$$

$$\int_0^a x^2 \int_y^a y^2 \; x \; (dx\,dy)$$

$$\int_0^1 x^2 \int_1^1 y^2 \; \cancel{d}$$

$$\int_0^1 x^2 \int_2^3 \frac{x}{x^2+y^2}$$

x value is 2

$$\frac{\cancel{x}}{}$$

$$\frac{2}{2^2+y^2}$$

$$\frac{2}{2^2+y^2}$$

$$\frac{\cancel{2}\,1}{\cancel{4}_2+y^2}$$

$$\frac{1}{2+y^2}$$

$$\int_0^a \int_y^a 2y^2$$

//ans

Name: R. Gokul

Reg. No: U22CN202.

Section: 'K'

Sub: ACCA.

Date: 25.4.2023.

8h 25/04/23

4/30

Part - B

6)
a) $x=0, y=1$ and $y=x$.

$y=1$ and $x=0=0$.

$y=x^2$ and $x=y2x+3$,

$Q=xyz$ and $r=\frac{\vec{r}}{r}$.

$dxdy = \int e^{-x2} dx$.

$\frac{\sin x}{x} dy\, dx$.

$x=0, =1, x\ y=0.\ (1,-2,-1)$

$(x+y)\ dx\ dy$.

$x^2+y^2+z^2+2xy=0$

$\frac{2xy}{2xy}=2+1=3$.

$y=2x+3=0.\ 2x$.

$x2+y2-2+22$.

$2+2+y+x^2=3$.

$(x+y)\ \cancel{dx}\ dy$.

$\frac{\sin x}{x} dy\ dx$.

4

$dxdy = \int e^{-x2} dx$.

$(x+y)dx-dy=0$.

CLA-II

Name: Anu Tayal. V
Register no: V22BM007
subject : Advanced calculus and complex Analysis
subject code: V20MABT02
Date : 31/5/23
course and sec : BME (B1)

SET-B

80/30

V. Jeya

PART-C (1x12=12)

8.a) Verify green's theorem in a plane for the integral $\int_C (xy+y^2)dx + x^2dy$ where c is closed curve of region bounded by $y=x$ and $y=x^2$.

Soln: Given:

$$\int_C (xy+y^2)\,dx + x^2\,dy$$

region bounded by $y=x$ and $y=x^2$

As we known that by green's theorem;

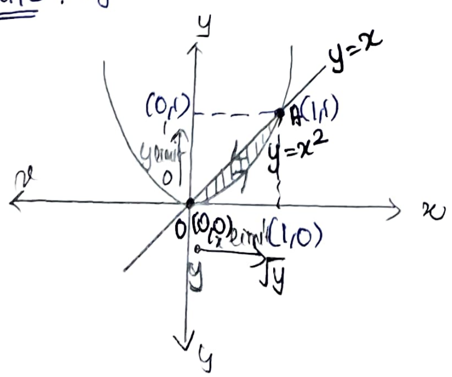
$$\int_C u\,dx + v\,dy = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

Comparing LHS and given integral.

$$\int_C u\,dx + v\,dy = \int_C (xy+y^2)\,dx + x^2\,dy$$

$$\Rightarrow u = (xy+y^2) \qquad v = x^2$$

RHS: (given boundaries)



solving both equation's

$y = x^2$ and $y = x$

$x = x^2$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0 ; x = 1$

when $x = 0 \Rightarrow y = (0)^2 = 0$ and $y = 0$

$x = 1 \Rightarrow y = (1)^2 = 1$ and $y = 1$

RHS:

$$\Rightarrow \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy \rightarrow (1)$$

$u = xy + y^2 \quad v = x^2$

$$\frac{\partial u}{\partial y} = x + 2y \quad \frac{\partial v}{\partial x} = 2x \rightarrow (3)$$

$\hookrightarrow (2)$

By (1) (2) & (3)

$$\Rightarrow \iint_S (2x - x - 2y)\,dx\,dy$$

$$= \iint_S (x - 2y)\,dx\,dy$$

we have variable function

$\Rightarrow$ inner limit variable

$\Rightarrow$ outer limit constant

limit of $x \Rightarrow y$ to $\sqrt{y}$

limit of $y \Rightarrow 0$ to 1

$$= \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left(\left[\frac{x^2}{2} - 2xy \right]_y^{\sqrt{y}} \right) dy$$

$$= \int_0^1 \left[\left(\frac{(\sqrt{y})^2}{2} - 2\sqrt{y}\cdot y\right) - \left(\frac{y^2}{2} - 2y^2\right)\right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - \frac{2y^{3/2}}{\cancel{2}} - \frac{y^2}{2} + 2y^2\right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} - \frac{y^2 + 4y^2}{2}\right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} + \frac{3y^2}{2}\right] dy$$

$$= \left[\frac{y^2}{4} - 2\,\frac{y^{5/2}}{5/2} + \frac{3}{2}\times\frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1(1)^2}{4} - \frac{4}{\cancel{3}}\times\frac{2(1)^{5/2}}{5} + \frac{3}{2}\times\frac{(1)^3}{3} - 0\right]$$

$$= \cancel{\frac{1}{4} - \frac{8}{15} + \frac{1}{2}} \qquad = \frac{1}{4} - \frac{4}{5} + \frac{1}{2}$$

$$= \cancel{\frac{15 - 8\times 4 + 30}{60}} \qquad = \frac{5 - 16 + 10}{20}$$

$$= \cancel{\frac{15 - 32 + 30}{60}} \qquad = \frac{15 - 16}{20}$$

$$= \frac{-1}{20}$$

$\neq$

$\frac{1}{2}+1$
$\frac{3}{2}$
$-\frac{1}{2}+2$
$\frac{-1+4}{2}$
$+\frac{3}{2}-\frac{1}{2}$
$\frac{2}{2}$
$\frac{1}{2}$
$y^{1/2}\cdot y^{1}$
$y^{1/2+1}$ $\frac{3}{2}+1$
$y^{3/2}$

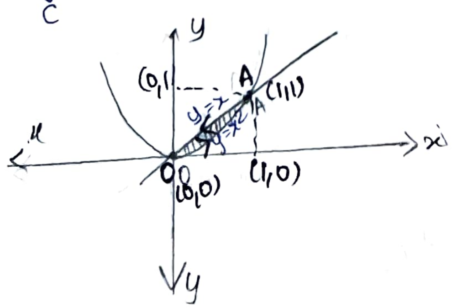
$y^{1/2+1}$
$y^{\frac{1}{2}+1}$
$\frac{1+2}{2} = \frac{3}{2}$

2 | 4, 5, 2
2 | 2, 5, 1
1, 5, 1

-32
$+30$
-2 15

LHS:

$$\int_C u\,dx + v\,dy$$



$$\int_C u\,dx + v\,dy = \int_{OA} + \int_{AO}$$

Along OA ; $y = x^2 \Rightarrow dy = 2x\,dx$; limit of x is from 0 to 1. $\rightarrow$ (4)

$$\int_C (xy + y^2)\,dx + x^2\,dy \rightarrow (5)$$

Put (4) in (5)

$$= \int_C \left(x(x^2) + (x^2)^2\right)dx + x^2(2x)\,dx$$

$$= \int_{x=0}^{x=1} (x^3 + x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (2x^3 + x^3 + x^4)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \left[\frac{3}{4} + \frac{1}{5} - 0\right]$$

$$= \frac{15+4}{20}$$

$$= \frac{19}{20} \rightarrow (6)$$

Along AO line $\Rightarrow y = x \Rightarrow dy = dx \Rightarrow$ limit of x is 1 to 0 ie A.C.W $\longrightarrow$ (7)

Put (7) in (5)

$$= \int_C (x(x) + x^2)\,dx + x^2\,dx$$

$$= \int (x^2 + x^2)\,dx + x^2 dx$$

$$= \int_{x=1}^{x=0} 3x^2\,dx$$

$$= \int_1^0 3x^2\,dx$$

$$= \left[\frac{3x^3}{3}\right]_1^0$$

$$= [x^3]_1^0$$

$$= 0 - 1$$

$$= -1 \quad \rightarrow (8)$$

Adding (6) and (8)

$$\int_{OA} + \int_{AO} = \frac{19}{20} - 1$$

12

$$= \frac{19-20}{20} = -\frac{1}{20}$$

LHS = RHS

Hence green's theorem verified.

PART-B $(2 \times 4 = 8)$

6a) find $L[e^{-3t} - 2\cos 2t + 3\sinh 3t + 5]$

as $L[e^{-at}] = \frac{1}{s+a}$ $\quad L[\cos at] = \frac{s}{s^2+a^2}$ $\quad L[\sinh at] = \frac{a}{s^2-a^2}$

$$L[1] = \frac{1}{s}$$

$$= L[e^{-3t}] - 2L[\cos 2t] + 3L[\sinh 3t] + 5L[1]$$

$$= \frac{1}{s+3} - 2 \times \frac{s}{s^2+4} + 3 \times \frac{3}{s^2-9} + 5 \times \frac{1}{s}$$

$= \frac{1}{S+3} - \frac{2S}{S^2+4} + \frac{9}{S^2-9} + \frac{5}{S}$

7. a) find $L[e^{3t}\sin 2t]$

$L[f(t)] = L[\sin 2t]$ as $L[\sin at] = \frac{a}{s^2+a^2}$

$= \frac{2}{s^2+2^2}$ $\sin at = \frac{a}{s^2+a^2}$

$= \frac{2}{s^2+4} = F(S)$ $\frac{2}{s^2+24}$

$e^{at} = (S-a)$

then

$L[e^{at}F(t)] = F(S-a)$ $\frac{2}{(S-3)^2+4} = \frac{2}{s^2-6s+9+4}$

$L[e^{3t}\sin 2t] = \left(\frac{2}{s^2+4}\right)$

$S \to S-a = S-3$

$= \frac{2}{(S-3)^2+4}$

$= \frac{2}{s^2+9-6s+4}$

$= \frac{2}{s^2-6s+13} = \frac{2}{s^2-6s+13} = \frac{2}{s^2-6s+13}$

PART-A $(5\times 2 = 10)$

1. Gauss divergence theorem:-

If $\vec{F}$ is vector point, finite, differentiable function in the region R enclosed or bound by the surface S then the surface integral of normal component of $\vec{F}$ over S is equal to integral of Divergence of $\vec{F}$ over V

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$ where $dv = dx\,dy\,dz$ for (3D)

2. Prove that $L[e^{-at}] = \frac{1}{s+a}$

To prove:

$L[e^{-at}] = \frac{1}{s+a}$

we know that

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$\Rightarrow L[e^{-at}] = \int_0^\infty (e^{-st} \cdot e^{-at})\,dt$

$= \int_0^\infty e^{(-st-at)}\,dt$

$= \int_0^\infty e^{-t(s+a)}\,dt$

as $\int e^{ax} dx = \frac{e^{ax}}{a}$

$= \frac{\left[e^{-t(s+a)}\right]_0^\infty}{-(s+a)}$

$= \frac{1}{-(s+a)} (e^{-\infty} - e^{0})$ $\quad \because e^{-\infty} = 0,\ e^{0} = 1$

$= \frac{1}{-(s+a)} (0-1)$

$= \frac{1}{s+a}$

hence proved.

3. Prove that $L[\sinh at] = \frac{a}{s^2-a^2}$

To prove

$L[\sinh at] = \frac{a}{s^2-a^2}$

we know that

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$.

$\sinh at = \frac{e^{\theta} - e^{-\theta}}{2}$

$L[\sinh at] = \frac{1}{2} L[e^{+\theta} - e^{-\theta}]$

$= \frac{1}{2}\left[L[e^{+\theta}] - L[e^{-\theta}]\right]$

here $\theta = at$

$= \frac{1}{2}\left[L[e^{+at}] - L[e^{-at}]\right]$

as we known

$L[e^{-at}] = \frac{1}{s+a}$ and $L[e^{at}] = \frac{1}{s-a}$

$= \frac{1}{2}\left[-\frac{1}{s+a} + \frac{1}{s-a}\right]$

$= \frac{1}{2}\left[\frac{(s-a) + (s+a)}{s^2 - a^2}\right]$

$= \frac{1}{2}\left[\frac{-s+a+s+a}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$

$= \frac{a}{s^2-a^2}$

hence proved.

---

4. Find $L[1] = \frac{1}{s}$

to find $L[1] = \frac{1}{s}$

we known that

$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$

$L[1] = \int_0^\infty e^{-st} (1)\,dt = \int_0^\infty e^{-st}\,dt$ as $\int e^{at}\,dt = \frac{e^{at}}{a}$ i

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{-s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-s}\left[0 - 1\right]$$

$$= \frac{1}{s}$$

hence proved.

5. Find $L[\cos 4t \cos 2t]$

as $\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$

$$\Rightarrow \cos 4t \cos 2t = \frac{\cos(4t+2t) + \cos(4t-2t)}{2}$$

$$= \frac{\cos(6t) + \cos 2t}{2}$$

$$L[\cos 4t \cos 2t] = \frac{1}{2} L[\cos(6t) + \cos(2t)]$$

$$= \frac{1}{2}\left[L[\cos(6t)] + L[\cos(2t)]\right]$$

as $L[\cos at] = \frac{s}{s^2+a^2}$

$$= \frac{1}{2}\left[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+36} + \frac{s}{s^2+4}\right]$$

$$= \frac{s}{2}\left[\frac{1}{s^2+36} + \frac{1}{s^2+4}\right]$$

$\cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$

$\frac{1(\cos 6t) + \cos(2t)}{2}$

$\cos at = \frac{s}{s^2+a^2}$

$= \frac{1}{2}\left[\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+36} + \frac{s}{s^2+4}\right]$

$e^{-3t} - 2\cos 2t + 3\sinh 3t + 5$

$\frac{1}{s+3} - 2\times\frac{s}{s^2+4} + 3\times\frac{3}{s^2-3^2} + \frac{5}{s}$

$\frac{1}{s+3} - \frac{2s}{s^2+4} + \frac{9}{s^2-9} + \frac{5}{s}$

$e^{3t}\sin 2t$

$\sin 2t = \frac{2}{s^2+4}$

$e^{at} = s \to s-a$

$= \frac{2}{(s-3)^2+4} = \frac{2}{s^2-6s+9+4}$

Bi-section

07

$s^2 - 6s + 12$ 12 34 12 1

$-b \pm \sqrt{\ }$

2

Name: V.Harishankar

Reg no: U22CC018

Section: CSE - R

Subject Code: U20MABT02

Subject: Advanced calculus and Complex Analysis

Date: 31.05.2023

SET-B

19/30

Part - A

1. Gauss Divergence Theorem.

$$\iint_R \bar{F}\cdot\hat{n}\,dy = \iiint_S \bar{\nabla}\cdot\bar{F}\,dv$$

$$\bar{F} = (x^2 - yz)\,\bar{i} + (y^2 - xz)\,\bar{j} + (z^2 - xy)\,\bar{k}$$

2. Sol: given

$$L[e^{-at}] = \frac{1}{s+a}$$

By the definition

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= \cancel{\tfrac{1}{2}} \frac{1}{-(S+a)} \left[e^{-\infty} - e^{-0}\right]$$

$$= \frac{1}{-(S+a)} \left[0-1\right]$$

$$L\left[e^{-at}\right] = \frac{1}{S+a}$$

Hence the proved

3. Sol: given

$$L\left[\sinh at\right] = \frac{a}{S^2-a^2}$$

By the defintion

$$L\left[f(t)\right] = \int_0^\infty e^{-St} f(t)\, dt$$

$$L\left[\sinh at\right] = \left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \tfrac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{(S-a)} - \frac{1}{(S+a)}\right]$$

$$= \tfrac{1}{2} \frac{(S+a) - (S-a)}{(S-a)-(S+a)}$$

$$= \tfrac{1}{\cancel{2}} \frac{\cancel{2}a}{S^2-a^2}$$

$$= \frac{a}{S^2-a^2}$$

Hence proved

4\. Sol: given

$$L[1] = 1/s$$

By the defination

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[1] = \int_0^\infty e^{-st} \, 1 \, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$L[1] = 1/s$$

Hence proved

5\. Sol:

$$L[\cos 4t \cos 2t] = L\left[\frac{\cos 6t + \cos 2t}{2}\right]$$

$$= \frac{1}{2} L[\cos 6t] + L[\cos 2t]$$

$$= \frac{1}{2}\left(\frac{s}{s^2+36} + \frac{s}{s^2+4}\right)$$

Part - B

7\. a)

Sol: given

$L[e^{3t} \sin 2t]$

$L[e^{3t} \sin 2t] = 1 e^{-3t}$

$= \frac{1}{s^2-4}$

$L\{e^{3t} \sin 2t\} = \frac{1}{s^2-4}$

$= \frac{1}{s^2+9-6s+4}$

$= \frac{1}{s^2+13-6s}$

$= \frac{1}{s^2-6s+13}$

6\. a)

Sol: given

$L[e^{-3t} - 2\cos 2t + 3\sin 3t + 5]$

$= L(e^{3t}) - 2t(\cos 2t) + 3t(\sin t) + 5$

2

$= \frac{1}{s-2} - \frac{2c}{s^2-4} + \frac{\cancel{3c}\ 3\ 9}{\cancel{s^3}+(s^2-9)} + 5/s$

Part - C

8. a)

Sol:

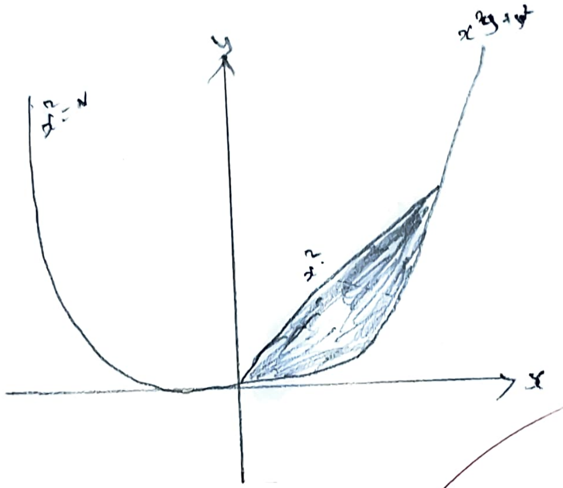
given integral

$$\int_C (xy+y^2)\, dx + x^2\, dy \quad \rightarrow (1)$$

$$\int_R M dx + N dy = \iint_C \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

$M = xy + y^2$ Pdw to x $\qquad$ $N = x^2$ Pdw to y

$\frac{\partial N}{\partial x} = x^2$, $\qquad \frac{\partial M}{\partial y} = xy + y^2$



Name: Mohammad Nadeem
Reg no:- U22BM066
Branch: Biomedical B1
Subject: ACCA
Set - B
CIA-2 Term 2.

CIA-2

7/30

Part-C

8)b) To verify Gauss divergence Theorem:

Given $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$

$x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c.$

To prove

LHS = RHS by Gauss divergence.

$\iint F\hat{n}\,ds = \iiint \operatorname{div}\vec{F}\,dv.$

RHS

$\iiint \operatorname{div}\vec{F}\,dv$

$\vec{F} = \nabla\cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$= \frac{\partial}{\partial x}(x^2-y) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$

$= \frac{x^2}{2} + yx + zx \qquad \nabla F = 2(x+y+z)$

$= \int_0^a\int_0^b\int_0^c \left(\frac{x^2}{2}+yx+zx\right)dx\,dy\,dz.$

RHS

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dv = \int_0^a \int_0^b \int_0^c 2(x+y+z)\, dx\, dy\, dz$$

$$= \iiint_{0\,0\,0} \left(\frac{x^2}{2} + yx + zx \right)$$

$$= \int_0^c \int_0^b \left(\frac{a^2}{2} + ya + za \right) dy\, dz$$

$$= \int \left[\frac{a^2 y}{2} + \frac{y^2 a}{2} \right)_0 dz$$

$$= \cancel{\int \left[\frac{a^2 b}{c} + \frac{ab^2}{2} + abc \right] dz}$$

$$\left[\frac{a^2 b}{2} + \frac{ab^2}{2} + \frac{abc}{2} \right]_0^c$$

$$\iiint \nabla \vec{F}\, dv = abc(a+b+c) \quad \text{— (1)}.$$

$$= abc(a+b+c) \doteq \text{RHS}.$$

LHS

$$\iint \vec{F} n\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$\int_0^c\int_0^b (x^2-y^2)dydz$$

$$\int_0^c\int_0^b \left(x^2y-\frac{y^2}{2}+dz\right)_0^b$$

$$\int_0^c \left(x^2y-\frac{y^2}{2}+dz\right)$$

$$\int_0^c \left(a^2bdz-\frac{b^2}{2}dz\right)$$

$$\left[a^2bz-\frac{b^2}{2}dz\right)$$

$$\left[a^2bz-\frac{b^2}{2}\frac{x^2}{2}\right]_0^c$$

$$a^2bc-\frac{b^2}{2}\frac{x^2}{2}\Big]_0^c$$

$$\iint_{S_1} a^2bc-\frac{b^2c^2}{a}$$

$$\iiint \vec{F}.n.ds=\iint (x^2-y)\vec{i}+$$

$$(y^2-x0)\vec{j}+(2^a-xy)\vec{k}(-i)\,dy\,dx$$

$$+\int_0^a\int_0^b (x^2-yz)\,dy\,dz$$

$$\iint_{0} F\,nds=\frac{b^2c^2}{y}.$$

LHS = RHS.

$abc[a+b+c] = abc[a+b+c]$.

Part - B

8) (a) $L[e^{-3t} - 2\cos 2t + 3\sinh 3t + 5]$

$L[e^{-3t}] - L[2\cos 2t] + L[3\sinh 3t] + L[5]$

$L[e^{-3t}] - \frac{6}{a^2 - s^2} + \frac{s^2}{a - s} + L(5)$..

Continuous Learning Assesment-3

Name: Anu Tayal
Register no: U22BM007
Subject: Advanced calculus and complex Analysis
Subject code: U20MABT02
Date: 23/6/23
Sec: Bio-A (B1)

SET-B

80/80

PART-C

a) find $f(z)$ if real part is $e^x(x\cos y - y\sin y)$ Also find conjugate.

Given:

$$R.P \Rightarrow u = e^x(x\cos y - y\sin y)$$
$$= e^x x\cos y - e^x y\sin y$$

To find:

$f(z)$ and conjugate.

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$u_x(x,y) = \cos y\left(e^x x + e^x(1)\right) - e^x y\sin y$$
$$= e^x x\cos y + e^x\cos y - e^x y\sin y$$

replace;
$x \to z \quad y \to 0$

$$u_x(z,0) = (\cos 0)e^z . z + e^z\cos(0) - e^z(0)\sin 0$$
$$= e^z . z + e^z = \phi_1(z,0)$$

12

$$u_y(x,y) = e^x x(-\sin y) - e^x\left(y(\cos y) + \sin y(1)\right)$$
$$= -e^x x\sin y - e^x y\cos y - e^x\sin y$$

replace $x \to z \quad y \to 0$

$$u_y(z,0) = -e^z . z\sin 0 - e^z(0)\cos(0) - e^z\sin 0$$
$$= 0 - 0 - 0$$
$$= 0 = \phi_2(z,0)$$

By milne thomsan method we know;
if Real part is given

$$F(z) = \int(\phi_1(z,0) - i\phi_2(z,0))dz$$

$$= \int[e^z(z+1)) - i(0)]dz$$

$$= \int(e^z(z+1))dz$$

$\int uv = uv_1 - u'v_2 + u''v_3$

$$= (z+1)e^z - 1e^z + 0(e^z)$$

here $u = z+1$ $v_1 = e^z$
$u' = 1$ $v_2 = e^z$
$u'' = 0$ $v_3 = e^z$

$$= e^z z + \cancel{e^z} - \cancel{e^z}$$

$$f(z) = e^z . z$$

To find conjugate.

$$f(z) = u + iv$$

$$z = x + iy$$

$$u + iv = e^{x+iy} . (x+iy)$$

$e^{a+m} = e^a . e^m$

$$= e^x . e^{iy} (x+iy)$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= e^x(\cos\theta + i\sin\theta)(x+iy)$$

$$= e^x(x\cos\theta + ix\sin\theta + iy\cos\theta + i^2y\sin\theta)$$

$$= e^x x\cos\theta + ie^x x\sin\theta + ie^x y\cos\theta - e^x y\sin\theta$$

$$u + iv = e^x(x\cos\theta - y\sin\theta) + i(e^x x\sin\theta + e^x y\cos\theta)$$

comparing on both sides
The conjugate of the real part is imaginary part which is $e^x(x\sin\theta + y\cos\theta)$.

PART-B

6(a) bilinear transformation that maps the point $z = 0, -i, -1$ onto the points $w = i, 1, 0$ respectively.

given:

$z_1 = 0 \quad z_2 = -i \quad z_3 = -1$

$w_1 = i \quad w_2 = 1 \quad w_3 = 0$

bilinear transformation;

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i - (-1))}{(0 - (-i))(-1 - z)}$$

$$\Rightarrow \cancel{-}\frac{(w - i)(1)}{(i - 1)\,w} = \cancel{-}\frac{z(-i + 1)}{i(z + 1)}$$

$$\Rightarrow \frac{(w - i)}{(wi - w)} = \frac{-iz + z}{iz + z}$$

$$\Rightarrow (iz + z)(w - i) = (-iz + z)(wi - w)$$

$$\Rightarrow iwz + wz - i^2 z - iz = -i^2 wz + iwz + iwz - wz$$

$$\Rightarrow iwz + wz - i^2 z - iz + i^2 wz - iwz - iwz + wz = 0$$

$$\Rightarrow \cancel{iwz} + wz - i^2 z - iz + i^2 wz - \cancel{iwz} - iwz + wz = 0$$

$$\Rightarrow wz + z - iz - \cancel{wz} - iwz + \cancel{wz} = 0$$

$$\Rightarrow wz - iwz + z(1 - i) = 0$$

$$\Rightarrow wz(1 - i) + z(1 - i) = 0$$

$$\Rightarrow (1 - i)(wz + z) = 0$$

$$\Rightarrow wz(1 - i) = -z(1 - i)$$

$$\Rightarrow w = \frac{-z(1 - i)}{z(1 - i)} \times \frac{(1 - i)}{(1 - i)} = \frac{-z(1^2 - i^2)}{z(1 - i^2)} = \frac{2z}{z(1-i)}$$

6b) construct analytic function $f(z)$ which real part $e^{x}\cos y$

given.

$$U = e^{x}\cos y$$

To find $f(z)$

$$U_x(x,y) = e^{x}\cos y$$

replace $x \to z$ $y \to 0$

$$U_x(z,0) = e^{z}\cos(0)$$

$$= e^{z} \times 1 \to \phi_1(z,0)$$

$$U_y(x,y) = e^{x}(-\sin y)$$

$x \to z$ $y \to 0$ $= -e^{x}\sin y$

$$U_y(z,0) = -e^{z}\sin(0) = 0 = \phi_2(z,0)$$

By milne thomsan method

$$f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))dz$$

$$= \int(e^{z} - i(0))dz$$

$$f(z) = \int e^{z}dz$$

$$= e^{z}$$

The analytic whose real part is $e^{x}\cos y$

$$u+iv = e^{x+iy}$$

$$= e^{x}.e^{iy}$$

$$= e^{x}(\cos y + i\sin y)$$

$$= e^{x}\cos y + ie^{x}\sin y$$

The imaginary Part is $e^{x}\sin y$.

(10)

1) Evaluate $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)} dz$ where C is the circle $|z|=1$

Using Cauchy integral formula.

Given:

$$\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)} dz$$

We know that

$$\int_C \frac{dz}{(z-z_0)} dz = 2\pi i f(z_0)$$

$$\frac{4z^2-4z+1}{(z^2+4)(z-2)} = \frac{Az+B}{(z^2+4)} + \frac{C}{(z-2)}$$

$$\frac{4z^2-4z+1}{(z^2+4)\cancel{(z-2)}} = \frac{(Az+B)(z-2)+C(z^2+4)}{(z^2+4)\cancel{(z-2)}}$$

$$4z^2-4z+1 = Az^2-2Az+Bz-2B+z^2C+4C$$

Comparing on both sides.

$A+C=4 \qquad -2A+B=-4 \qquad -2B+4C=1$

$A=4-C \Rightarrow -2(4-C)+B=-4$

$\Rightarrow -8+2C+B=-4$

$\Rightarrow 2C+B=-4+8$

$\Rightarrow 2C+B=4 \times 2 \qquad -2B+4C=1$

$\Rightarrow 4C+2B=8 \to ① \qquad -2B+4C=1 \to ②$

add ① and ②

$4C+4C=8+1$

$8C=9$

$C=9/8$

$\Rightarrow A=4-C=4-\frac{9}{8}=\frac{32-9}{8}=\frac{23}{8}$

$\overset{2}{\cancel{32}}$
-9
23

$\Rightarrow \cancel{A} \times \frac{9}{\cancel{8}\,4} + B = 4$

$B = 4 - \frac{9}{4}$

$= \frac{36-9}{4} = \frac{27}{4}$

$$\Rightarrow \int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)}dz = \int_C \frac{\frac{23}{8}z + \frac{27}{4}}{(z^2+4)} dz + \int_C \frac{9/8}{(z-2)}$$

$$= \int_C \frac{\frac{23}{8}z + \frac{27}{4}}{(z^2+4)} dz + \int_C \frac{9/8}{z-2}$$

comparing RHS with Cauchy integral formula.

$\Rightarrow \quad z^2+4 = z - z_0 = 0 \qquad z-2=0$

$\Rightarrow z^2 = -4 \qquad z = 2$

$\Rightarrow \sqrt{z^2} = \sqrt{-4}$

$= z = \pm i2$

$\Rightarrow z_0 = +i2, -i2, 2$

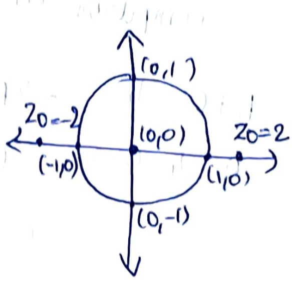
Given region / circle $|z| = 1$

$|x+iy| = 1$

$\sqrt{x^2 + i^2y^2} = 1$

$\sqrt{x^2+y^2} = 1$

$x^2+y^2 = 1$



As the points are outsides the circle.

hence $\int_C \frac{4z^2-4z+1}{(z^2+4)(z-2)} dz = 0.$

10

PART-A

1. find fixed points of transformation

$$w = \frac{6z-9}{z}$$

$$z = \frac{6z-9}{z}$$

$$z^2 - 6z + 9 = 0$$

$$(z-3)(z-3) = 0$$

$$z = 3, 3$$

9

-3 -3

$z = 3, 3$ are the fixed of the transformation $w = \frac{6z-9}{z}$

2. Poles of function $f(z) = \frac{z}{(z+1)(z-i)}$

given:

$$f(z) = \frac{z}{(z+1)(z-i)}$$

to find singularity;

$(z+1) = 0)$ & $(z-i = 0)$

$z = -1$ $z = i$

$z = -1, i$ are singular points or pole of order 1 or simple poles.

3. find residue at $z=0$ for $f(z) = \cot z$

$\int f(z)\,dz$

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

$z=0, \pm\pi, \pm 2\pi, \ldots$ $\pm n\pi$

$\sin z = 0$

$\Rightarrow z = 0$ is a singular pole or point or simple pole

$\Rightarrow \left[\text{Residue } F(z)\right]_{z=a} = \lim_{z \to a} (z-a) f(z)$ (Residue about simple pole)

~~$f(z) = f'(z) \times f''(z)$~~

$\Rightarrow \left[\text{Residue } F(z)\right]_{z=0} = \lim_{z \to 0} (z-0) \times \cot z$

0 ~~f(z)~~ cot 0

0 $\frac{1}{\sqrt{3}}$ 1 $\sqrt{3}$ ∞

0

$$= \left[\underset{z=0}{\text{residue}}\ F(z)\right] = \lim_{z\to 0} \frac{\cos z}{\sin z} \times z$$

$$= \lim_{z\to 0} \frac{\cos(0)}{\sin(0)} \times 0$$

$$= \infty \Rightarrow \frac{1}{0} \text{ form}$$

$\frac{f(z)}{g(z)} = \frac{\cos z}{\sin z}$

$\frac{\cos z}{\cos}$

$\frac{f'(z)}{g'(z)} = 1$

$z \to 0$

$\frac{d(\sin z)}{dz} = \cos z$

hence ; (or)

$$\Rightarrow \left[\underset{z=0}{\text{residue}}\ F(z)\right] = \lim_{z\to 0} \frac{\cos 0}{\sin 0} = \frac{1}{0} \text{ form} \Rightarrow$$

divide ~~multiply~~ by f'(denominator)

$$= \frac{\cos z}{\cos z} = \frac{\cos(0)}{\cos(0)} = 1$$

4. Classify singularity of function $f(z) = \frac{\sin z}{z}$

Given: $f(z) = \frac{\sin z}{z}$

Put $z=0$

$$\Rightarrow f(z) = \frac{\sin 0}{0} = \frac{0}{0} \text{ form.}$$

and multiply by f'(numerator) and divide ~~multiply~~ by f'(denominator)

$$\Rightarrow f(z) = \frac{\cos z}{1}$$

$\frac{d}{dz}(\sin z) = \cos z$

$\frac{d}{dz}(z) = 1$

Put $z=0$

$$\Rightarrow f(z) = \cos 0 = 1$$

$\Rightarrow$ Removable singularity.

5. Evaluate $f(z) = \frac{z}{(z+1)(z+2)}$ about the pole $z=-2$

Given:

$$f(z) = \frac{z}{(z+1)(z+2)}$$

To find singularity

$z+1=0$; $z+2=0$

$z=-1$; $z=-2$ are singular / simple poles of order 1.

Residue about pole $z=-2$ (simple pole)

$$\Rightarrow \left[\text{Residue } f(z)\right]_{z=a} = \lim_{z\to a} (z-a) f(z)$$

$$\Rightarrow \left[\text{Residue } f(z)\right]_{z=-2} = \lim_{z\to -2} (\cancel{z+2}) \left(\frac{z}{(z+1)\cancel{(z+2)}}\right)$$

$$= \lim_{z\to -2} \frac{z}{(z+1)}$$

$$\Rightarrow \left[\text{Residue } f(z)\right]_{z=-2} = \frac{-2}{-2+1} = \frac{-2}{-1} = 2.$$

4b) find residue of $f(z) = \frac{z^2}{(z-1)^2}$ where C is $|z|=3$

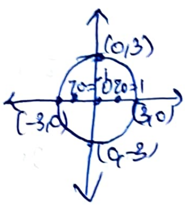
$$f(z) = \frac{z^2}{(z-1)(z+1)}$$

given : $|z| = 3$

$$|x+iy| = 3$$

$$\sqrt{x^2 - i^2y^2} = 3$$

$$\Rightarrow x^2 + y^2 = 3^2$$



$(z-1)^2 = 0$

$z-1=0$

$z=+1$

Error

$z-1=0 \qquad z+1=0$

$z=1 \qquad z=-1$

$z=1,-1$ are singular points of order 1.

$$\left[\text{Residue of } f(z)\right]_{z=1} = \lim_{z\to 1} (\cancel{z-1}) \frac{z^2}{\cancel{(z-1)}(z+1)}$$

$$= \lim_{z\to 1} \frac{z^2}{z+1}$$

$$= \frac{(1)}{1+1}$$

$$= \frac{1}{2}$$

$$\left[\text{Residue of } f(z)\right]_{z=-1} = \lim_{z\to -1} \frac{(z+1)\, z^2}{(z-1)\cancel{(z+1)}}$$

$$= \lim_{z\to -1} \frac{z^2}{z-1}$$

$$= \frac{(-1)^2}{-1-1} = \frac{1}{-2}$$

we know that

$$\int_C \frac{dz}{(z-z_0)} dz = 2\pi i\, f(z_0) \qquad \text{given } |z| = 3.$$

$\Rightarrow$ circle of radius 3

given $\dfrac{z^2}{(z-1)^2} = \dfrac{z^2}{(z-1)(z+1)} \Rightarrow f(z) = z^2$

$$\int_C \frac{z^2}{(z+1)(z-1)} dz$$

$$\frac{1}{(z+1)(z-1)} = \frac{A}{(z+1)} + \frac{B}{(z-1)}$$

$$= \frac{A(z-1) + B(z+1)}{(z+1)(z-1)}$$

$$1 = (z-1)A + B(z+1)$$

Put $z = 1$

$1 = 0(A) + 2B$

$B = 1/2$

Put $z = -1$

$1 = -2A + B(0)$

$A = -1/2$

$$\int_C \frac{z^2}{(z+1)(z-1)} dz = \frac{-1}{2}\int \frac{z^2}{(z+1)} + \frac{1}{2}\int \frac{z^2}{z-1}$$

$f(-1) = \dfrac{(-1)^2}{} = 1$

$f(1) = 1^2 = 1$

$z - z_0 = z+1 = 0$

$z = -1$

$z-1 = z - z_0 = 0$

$z = +1$

$z_0 = -1, 1$ both lie inside the circle.

$$\int_C \frac{z^2}{(z+1)(z-1)} dz = -\frac{1}{2} \times 2\pi i\, f(z_0) + \frac{1}{2} \times 2\pi i\, f(z_0)$$

$$= -\frac{1}{2} \times 2\pi i \times 1 + \frac{1}{2} \times 2\pi i \times 1$$

$$= -\pi i + \pi i$$

$$= 0.$$

NAME: K. DIYA REG.NO: U22BM025 'B1' sec

SUBJECT: ADVANCED CALCULUS AND COMPLEX ANALYSIS. U20MABT02

DATE: 23/6/23 CLA-III SET: B

20/30

PART-C

(8) a)

$e^x(x\cos y - y\sin y) = e^x x\cos y - e^x y\sin y$

diff. w.r.t x & y:

$u_x = (e^x\cos y + xe^x\cos y) - e^x y\sin y$

$v_y = -e^x x\sin y - e^x(y\cos y + \sin y)$

$v_y = -e^x x\sin y - e^x y\cos y - e^x\sin y$

Replace x by z, y by 0:

$u_x = e^z + ze^z - e^z(0) = e^z + ze^z = \phi_1(z,0)$

$v_y = -e^z z\sin(0) - e^z(0)\cos 0 - e^z\sin(0)$

$v_y = 0 = \phi_2(z,0)$

→ using Milne Thomson Method:

$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz$

$= \int(e^z + ze^z)dz$

$= \int e^z[1+z]dz \rightarrow uv' - vu'$

$= (z+1)e^z - 1.e^z$

$= (z+1)e^z - e^z$

$= ze^z + e^z - e^z = ze^z$

$\Rightarrow \boxed{F(z) = ze^z}$

To find conjucate, $f(z) = ze^z$

$\hookrightarrow u+iv = (x+iy)e^{x+iy}$

$(x+iy)e^x \cdot e^{iy} \Rightarrow (xe^x + iye^x)e^{iy}$

$u+iv = [xe^x + iye^x][\cos y + i\sin y]$

$\Rightarrow xe^x\cos y + xe^x i\sin y + iye^x\cos y + i^2 ye^x\sin y$

$xe^x\cos y + xe^x i\sin y + iye^x\cos y - ye^x\sin y$

$\Rightarrow (xe^x\cos y - ye^x\sin y) + i(xe^x\sin y + ye^x\cos y)$

$\therefore$ conjucate of real part is $i(xe^x\sin y + ye^x\cos y)$.

PART-A

(2) $f(z) = \dfrac{z}{(z+1)(z-i)}$

Put,

$z+1=0$ $\quad z-i=0$

$z=-1$ $\quad z=i$

$z=-1$, has a single pole $=1$

$z=i$, has a single pole $=1$

(5) $f(z) = \dfrac{z}{(z+1)(z+2)}$

$= \int_C \dfrac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$ $\quad r=1$.

(4) $f(z) = \frac{\sin z}{z}$

Put,

$z = 0 \rightarrow \frac{\sin z}{0} \Rightarrow \infty$

$\therefore$ No singularity for this function.

(3) $z = 0$

$f(z) = \cot z.$

$\rightarrow f(0) = \cot(0)$

$=$

PART-B

(7) (b) $f(z) = \frac{z^2}{(z-1)^2}$

$\rightarrow \int_C \frac{f(z)}{z - z_0} dz = \int_C \frac{z^2}{(z-1)^2}$

$(z-1)^2 = 0$

Sq. on both sides

$\Rightarrow z - 1 = 0$

$\boxed{z = 1}$

$\Rightarrow \boxed{z_0 = 1}$

$|z| = 3 \quad (r = 3)$

$|x + iy| = 3$

$\sqrt{x^2 + y^2} = 3$ (sq on both sides)

$\Rightarrow x^2 + y^2 = (3)^2 \rightarrow 9.$

To find z_0,

$2\pi i f(z_0)$

$\Rightarrow 2\pi i(1)$

$\Rightarrow 2\pi i$

(6)

$z = 0, -i, -1$ $\quad$ $w = i, 1, 0$

$z_1 \quad z_2 \quad z_3$ $\quad$ $w_1 \quad w_2, \quad w_3$

→ Bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\frac{w_1(w/w_1 - 1)(w_2 - w_3)}{w_1(1 - w_2/w_1)(w_3 - w)} = \frac{(z - z_1)\, z_2(1 - z_3/z_2)}{z_2(z_1/z_2 - 1)(z_3 - z)}$$

$z = 0, -i, -1$

$z_1 \quad z_2 \quad z_3$

$w = i, 1, 0$

$w_1 \quad w_2 \quad w_3$

Bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\frac{w_1(w/w_1 - 1)\, w_2(\ldots)}{w_1(\ldots)} = \frac{z_1(z/z_1 - 1)(z_2 - z_3)}{z_1(1 - z_2/z_1)(z_3 - z)}$$

$\frac{w - w_1}{w_1 - w_2}(-1) = (-1)\frac{z_2 - z_3}{z_3 - z}$

$\Rightarrow \frac{-w + i}{-i + 1} = \frac{i - 1}{1 + z}$

$-w + i =$

$f(z) = \frac{z^2}{(z-1)^2}$

$\int_C \frac{f(z)}{z - z_0}\, dz \Rightarrow$

$(z - 1)^2 = 0$

Sq. on both sides

$z - 1 = 0$

$z = 1$

$\Rightarrow \boxed{z_0 = 1}$

$f(z_0) = z^2$

$= (1)^2 = 1$

$z_0 = 1$

$(z-1)^2 = 0$

$\boxed{z = 1}$

$|z| = 3$

$\boxed{r = 3}$

$x^2 + y^2 = (3)^2$

$= 9$

To find z_0.

$2\pi i f(z_0)$

$\Rightarrow 2\pi i f(1)$

$2\pi i$

To find z_0

$2\pi i f(z_0)$

$2\pi i(9)$

$18\pi i$

$$\frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i + 1)}{(0 + i)(-1 - z)}$$

NAME : P Sampoorna Lakshmi

Register No: U22BM077

section: B1

sub : Advanced calculus and complex analysis

set : B

PART-C

8 (a) Given

$e^x(x\cos y - y\sin y)$

$e^x x\cos y - e^x y\sin y$

$V_y = e^x x(\sin y - \cos y(1)) - e^x y \cos y$

$V_y = e^x x(\sin y - \cos y) - e^x y\sin y$

$e^x x(e(x) - \cos e(x)^x) - e^x y\sin y$

According to millon thamson method

$e^x x(e^x - e(x)^x) - e^x y\sin y$

$f(z) = z^2 - z$

Find out conjugation

$f(z) = z^2 - z$

$f(z) = z^2 - z$

$f2 = z$

$(z^2 = i$

2/30

conjugation of the $f(z)$

the $f(z) = \bar{z}$

$\bar{z} = x\cos e^{x} - y\sin y$

$\bar{z} = x\cos e^{x} - y\sin y$

is the conjugation

$\bar{z} = z^{2} - 1$

the $\bar{z} = z^{2} - 1$

is

$e^{x}\cos x - e^{x} y\sin y$

$\int f(z) = \cancel{R}i = iz - z^{2}$

is the conjugated

$e^{x}(x\cos y - y\sin y)$ is verified.

Bharath
INSTITUTE OF HIGHER EDUCATION AND RESEARCH
Declared as **Deemed-to-be-University** u/s 3 of the UGC Act, 1956
173, Agaram Main Rd., Selaiyur, Chennai, Tamil Nadu 600073

www.bharathuniv.ac.in